MATH 100

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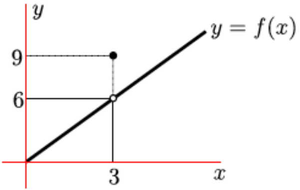
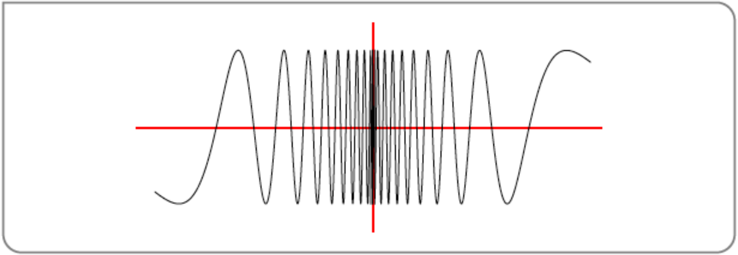
2019

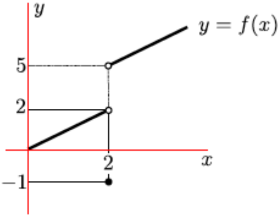
$$f(x) = \begin{cases} 2x & x < 3 \\ 9 & x = 3 \\ 2x & x > 3 \end{cases}$$



$f(x) = \sin(\frac{\pi}{x})$

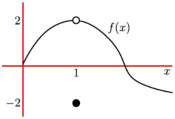


$$f(x) = \begin{cases} x & x < 2 \\ -1 & x = 2 \\ x + 3 & x > 2 \end{cases}$$



Example

Consider the graph of the function $f(x)$.



Then

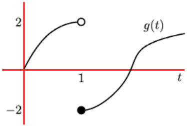
$$\lim_{x\to 1^-} f(x) =$$

$$\lim_{x\to 1^+} f(x) =$$

$$\lim_{x\to 1} f(x) =$$

Example

Consider the graph of the function $g(t)$.



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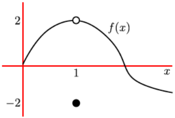

Then

$$\lim_{x\to 1^-} f(x) = 2$$

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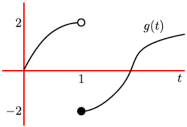
$$\lim_{x\to 1} f(x) = 2$$

Example

Consider the graph of the function $g(t)$.



Then

$$\lim_{t\to 1^-} g(t) = 2$$

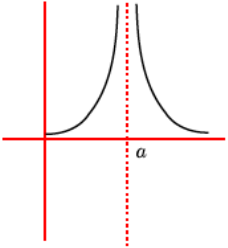
$$\lim_{t\to 1^+} g(t) = -2$$

$$\lim_{t\to 1} g(t) = DNE$$

When the limit goes to infinity

Example

Consider the graph for the function $f(x)$.



$$\lim_{x \to a} f(x) = +\infty$$

Example

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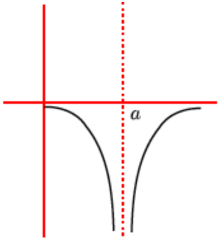
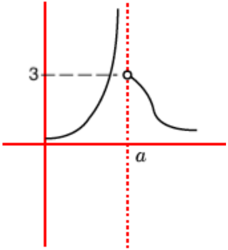


$$\lim_{x \to a} g(x) = -\infty$$

Example

Consider the graph for the function $h(x)$.

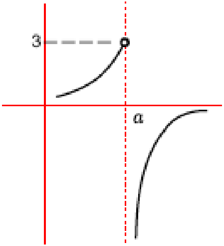


$$\lim_{x \to a^-} h(x) =$$

$$\lim_{x \to a^+} h(x) =$$

Example

Consider the graph for the function $s(x)$.

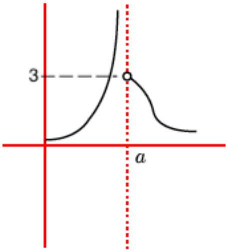


$$\lim_{x \to a^-} s(x) =$$

$$\lim_{x \to a^+} s(x) =$$

Example

Consider the graph for the function $h(x)$.

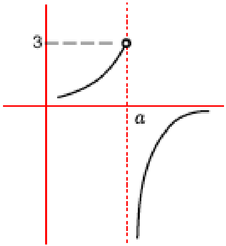


$$\lim_{x \to a^-} h(x) = +\infty$$

$$\lim_{x \to a^+} h(x) = 3$$

Example

Consider the graph for the function $s(x)$.



$$\lim_{x \to a^-} s(x) = 3$$

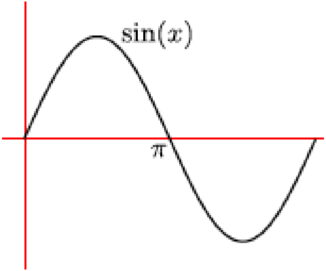
$$\lim_{x \to a^+} s(x) = -\infty$$

Example

Consider the function

$$g(x) = \frac{1}{sin(x)}.$$

Find the one-side limits of this function as $x \to \pi$.



$$\lim_{x \to \pi^-} \frac{1}{sin(x)} = +\infty$$

$$\lim_{x \to \pi^+} \frac{1}{sin(x)} = -\infty$$

Second Session Outline

- Arithmetic of the Limits
- Limit of a ratio: what will happen if the limit of the denominator is zero. For example,

$$\lim_{x\to 0} \frac{1}{x^2}? \quad \text{and} \quad \lim_{x\to 1} \frac{x^3 - x^2}{x-1} =?$$

- Sandwich/ Squeeze/Pinch Theorem
- limit at infinity

Arithmetic of the Limits

Theorem

Let $a, c \in \mathbb{R}$. The following two limits hold

$$\lim_{x \to a} c = c \qquad \lim_{x \to a} x = a$$

Example

$$\lim_{x \to 3} -2 = -2 \qquad \lim_{x \to -1} x = -1$$

Theorem

(**Arithmetic of Limits**) *Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x's that lie in some interval about a (but f and g need not to be defined exactly at a).*

$$\lim_{x \to a} f(x) = F \qquad \lim_{x \to a} g(x) = G$$

exists with $F, G \in \mathbb{R}$. Then the following limits hold

- $\lim_{x \to a}(f(x) + g(x)) = F + G$*–limit of the sum is the sum of the limits.*

Theorem

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- $\lim_{x\to a}(f(x) - g(x)) = F - G$*–limit of the difference is the difference of the limits.*

Theorem

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- $\lim\limits_{x \to a}(f(x) - g(x)) = F - G$*–limit of the difference is the difference of the limits.*
- $\lim\limits_{x \to a} cf(x) = cF$.

Theorem

(**Arithmetic of Limits**) *Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all x's that lie in some interval about a (but f and g need not to be defined exactly at a).*

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- $\lim_{x \to a}(f(x) - g(x)) = F - G$*–limit of the difference is the difference of the limits.*
- $\lim_{x \to a} cf(x) = cF$.
- $\lim_{x \to a}(f(x).g(x)) = F.G$*–limit of the product is the product of the limits.*

If $G \neq 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}$$

Example

Given

$$\lim_{x \to 1} f(x) = 3 \quad \text{and} \quad \lim_{x \to 1} g(x) = 2$$

We have

$$\lim_{x \to 1} 3f(x) =$$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) =$$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

$$\lim_{x\to 1} 3f(x) - g(x) = 3 \times \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x\to 1} f(x)g(x) =$$

Example

Given

$$\lim_{x\to1} f(x) = 3 \quad \text{and} \quad \lim_{x\to1} g(x) = 2$$

We have

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$$\lim_{x\to1} 3f(x) - g(x) = 3 \times \lim_{x\to1} f(x) - \lim_{x\to1} g(x) = 3 \times 3 - 2 = 7.$$

$$\lim_{x\to1} f(x)g(x) = \lim_{x\to1} f(x). \lim_{x\to1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x\to1} \frac{f(x)}{f(x) - g(x)} =$$

Example

Given

$$\lim_{x\to 1} f(x) = 3 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

We have

$$\lim_{x\to 1} 3f(x) = 3 \times \lim_{x\to 1} f(x) = 3 \times 3 = 9.$$

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$$\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} f(x). \lim_{x\to 1} g(x) = 3 \times 2 = 6.$$

$$\lim_{x\to 1} \frac{f(x)}{f(x) - g(x)} = \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} f(x) - \lim_{x\to 1} g(x)} = \frac{3}{3-2} = 3.$$

Example

$$\lim_{x \to 3} 4x^2 - 1 =$$

$$\lim_{x \to 2} \frac{x}{x-1} =$$

Example

$$\lim_{x \to 3} 4x^2 - 1 = 4 \times \lim_{x \to 3} x^2 - \lim_{x \to 3} 1 = 35.$$

$$\lim_{x \to 2} \frac{x}{x-1} = \frac{\lim_{x \to 2} x}{\lim_{x \to 2} x - \lim_{x \to 1} 1} = \frac{2}{2-1} = 2.$$

Limit of a ratio: what will happen if the limit of the denominator is zero.

Limit of a ratio: what will happen if the limit of denominator is zero:

- the limit does **not exist**, eg.

$$\lim_{x \to 0} \frac{x}{x^2} = \lim_{x \to 0} \frac{1}{x} = DNE$$

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$$\lim_{x\to 0}\frac{x}{x^2}=\lim_{x\to 0}\frac{1}{x}=DNE$$

– the **limit is** $\pm\infty$, eg.

$$\lim_{x\to 0}\frac{x^2}{x^4}=\lim_{x\to 0}\frac{1}{x^2}=+\infty \qquad \text{or} \qquad \lim_{x\to 0}\frac{-x^2}{x^4}=\lim_{x\to 0}\frac{-1}{x^2}=-\infty.$$

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- the **limit is** 0, eg.

$$\lim_{x\to 0}\frac{x^2}{x}=\lim_{x\to 0}x=0.$$

Limit of a ratio: what will happen if the limit of denominator is zero:

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- the **limit is** 0, eg.

$$\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0.$$

- the **limit exists and it nonzero**, eg.

$$\lim_{x\to 0} \frac{x}{x} = 1.$$

Theorem

Let n be a positive integer, let $a \in R$ and let f be a function so that

$$\lim_{x \to a} f(x) = F$$

for some real number F. Then the following holds

$$\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x) \right)^n = F^n$$

so that the limit of a power is the power of the limit.

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so that the limit of a power is the power of the limit. Similarly, if

- *n is an even number and $F > 0$, or*
- *n is an odd number and F is any real number*

then

$$\lim_{x \to a} (f(x))^{1/n} = \left(\lim_{x \to a} f(x) \right)^{1/n} = F^{1/n}.$$

Example

$$\lim_{x \to 4} x^{1/2} =$$

$$\lim_{x \to 4} (-x)^{1/2} =$$

$$\lim_{x \to 2} (4x^2 - 3)^{1/3} =$$

Example

$$\lim_{x \to 4} x^{1/2} = 4^{1/2} = 2.$$

$$\lim_{x \to 4} (-x)^{1/2} = -4^{1/2} = \text{not a real number.}$$

$$\lim_{x \to 2} (4x^2 - 3)^{1/3} = (4(2)^2 - 3)^{1/3} = (13)^{1/3}.$$

Limit of a ratio: what will happen if the limit of the numerator and denominator are zero, for example,

$$\lim_{x\to 1} \frac{x^3 - x^2}{x - 1} = ?$$

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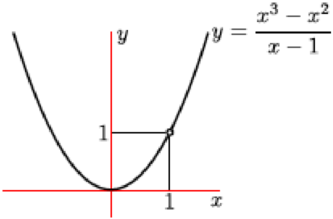


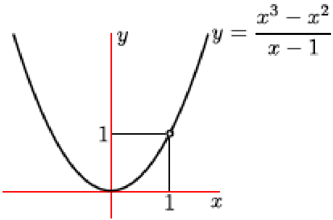
Theorem

If $f(x) = g(x)$ except when $x = a$ then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

provided the limit of g exists.

$$\frac{x^3 - x^2}{x - 1} = \begin{cases} x^2 & x \neq 1 \\ \text{undefined} & x = 1. \end{cases} \Rightarrow \lim_{x \to 1} \frac{x^3 - x^2}{x - 1} = \lim_{x \to 1} x^2 = 1.$$

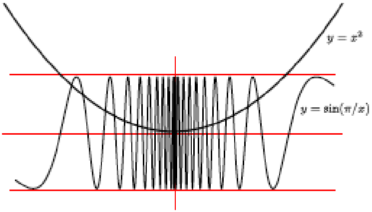


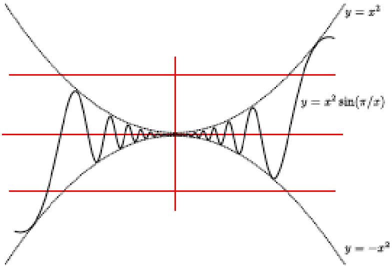
Sandwich/ Squeeze/Pinch Theorem

Example

Compute

$$\lim_{x \to 0} x^2 \sin(\frac{\pi}{x})$$





Example

Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \to 1} f(x)?$$

Example

Let $f(x)$ be a function such that $1 \leq f(x) \leq x^2 - 2x + 2$. What is

$$\lim_{x \to 1} f(x)?$$

Solution

Consider that

$$\lim_{x \to 1} x = 1 \qquad \textit{and} \qquad \lim_{x \to 1} x^2 - 2x + 2 = 1.$$

Therefore, by the sandwich/pinch/squeeze theorem

$$\lim_{x \to 1} f(x) = 1.$$

Example

We want to compute

$$\lim_{x\to+\infty}\frac{1}{x} \qquad \text{and} \qquad \lim_{x\to-\infty}\frac{1}{x}$$

By plug in some large numbers into $\frac{1}{x}$ we have

-10000	-1000	$-100\|\| \circ \|\|100$	1000	10000
-0.0001	-0.001	$-0.01\|\| \circ \|\|0.01$	0.001	0.0001

We see that as x is getting bigger and positive the function $\frac{1}{x}$ is getting closer to 0. Thus,

$$\lim_{x\to+\infty}\frac{1}{x}=0.$$

Moreover,

$$\lim_{x\to-\infty}\frac{1}{x}=0.$$

Limit at Infinity

Definition

(**Informal limit at infinity.**) We write

$$\lim_{x \to \infty} f(x) = L$$

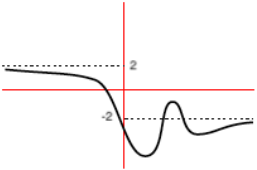
when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and positive.
Similarly, we write

$$\lim_{x \to -\infty} f(x) = L$$

when the value of the function $f(x)$ gets closer and closer to L as we make x larger and larger and negative.

Example

Consider the graph of the function $f(x)$.



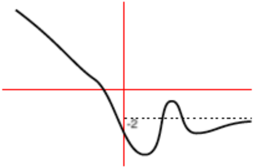
Then

$$\lim_{x\to\infty} f(x) =$$

$$\lim_{x\to-\infty} f(x) =$$

Example

Consider the graph of the function $g(x)$.



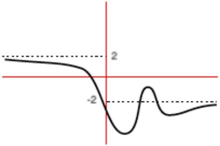
Then

$$\lim_{x\to\infty} g(x) =$$

$$\lim_{x\to-\infty} g(x) =$$

Example

Consider the graph of the function $f(x)$.



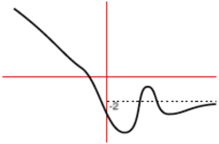
Then

$$\lim_{x \to \infty} f(x) = -2$$

$$\lim_{x \to -\infty} f(x) = 2$$

Example

Consider the graph of the function $g(x)$.



Then

$$\lim_{x \to \infty} g(x) = -2$$

$$\lim_{x \to -\infty} g(x) = +\infty$$

Review of the third session

Review

Theorem

sandwich (or squeeze or pinch) *Let* $a \in \mathbb{R}$ *and let* f, g, h *be three functions so that*

$$f(x) \leq g(x) \leq h(x)$$

for all x *in an interval around* a*, except possibly at* $x = a$*. Then if*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

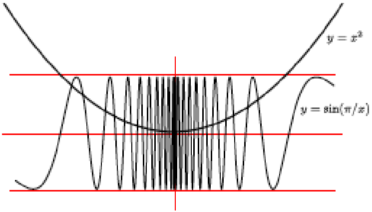
then it is also the case that

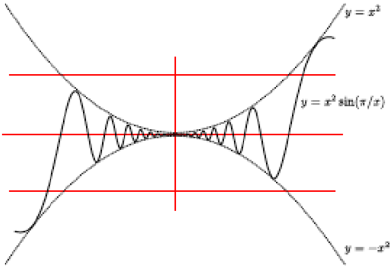
$$\lim_{x \to a} g(x) = L.$$

Example

Compute

$$\lim_{x \to 0} x^2 \sin(\frac{\pi}{x})$$





Theorem

Let $c \in \mathbb{R}$ then the following limits hold

$$\lim_{x \to +\infty} c = c \qquad \lim_{x \to -\infty} c = c$$

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

Outline For the Fourth Session

- Limit at Infinity

Limit at Infinity

Theorem

Let $f(x)$ and $g(x)$ be two functions for which the limits

$$\lim_{x \to \infty} f(x) = F \qquad \lim_{x \to \infty} = G$$

exist. Then the following limits hold

$$\lim_{x \to \infty} (f(x) + g(x)) = F \pm G$$

$$\lim_{x \to \infty} f(x)g(x) = FG$$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{F}{G} \quad \textit{provided } G \neq 0$$

and for rational numbers r,

$$\lim_{x\to\infty} (f(x))^r = F^r$$

provided that $f(x)^r$ is defined for all x.
The analogous results hold for limits to $-\infty$.

Warning: Consider that

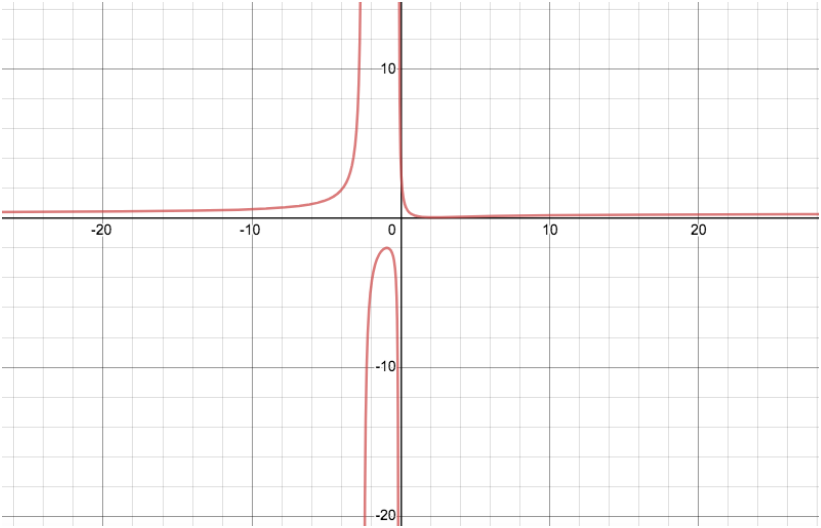
$$\lim_{x \to +\infty} \frac{1}{x^{1/2}} = 0$$

However,

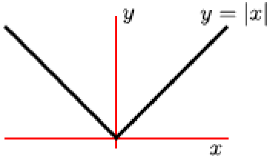
$$\lim_{x \to +\infty} \frac{1}{(-x)^{1/2}}$$

does not exist because $x^{1/2}$ is not defined for $x < 0$.

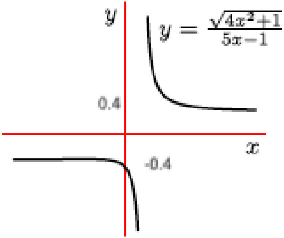
$$f(x) = \frac{x^2-3x+4}{3x^2+8x+1}$$



$$\sqrt{x^2} = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$



$$y = \frac{\sqrt{4x^2+1}}{5x-1}$$



Theorem

Let $a, c, H \in \mathbb{R}$ and let f, g, h be functions defined in an interval around a (but they need not be defined at $x = a$), so that

$$\lim_{x \to a} f(x) = +\infty \qquad \lim_{x \to a} g(x) = +\infty \qquad \lim_{x \to a} h(x) = H$$

1. $$\lim_{x \to a} (f(x) + g(x)) =$$
2. $$\lim_{x \to a} (f(x) + h(x)) =$$
3. $$\lim_{x \to a} (f(x) - g(x)) =$$
4. $$\lim_{x \to a} (f(x) - h(x)) =$$

Theorem

Let $a, c, H \in \mathbb{R}$ and let f, g, h be functions defined in an interval around a (but they need not be defined at $x = a$), so that

$$\lim_{x \to a} f(x) = +\infty \qquad \lim_{x \to a} g(x) = +\infty \qquad \lim_{x \to a} h(x) = H$$

1. $$\lim_{x \to a} (f(x) + g(x)) = +\infty.$$
2. $$\lim_{x \to a} (f(x) + h(x)) = +\infty.$$
3. $$\lim_{x \to a} (f(x) - g(x)) = \textit{undetermined.}$$
4. $$\lim_{x \to a} (f(x) - h(x)) = +\infty.$$

Theorem

5.

$$\lim_{x \to a} cf(x) = \begin{cases} & c > 0 \\ & c = 0 \\ & c < 0 \end{cases}$$

6.

$$\lim(f(x).g(x)) =$$

7.

$$\lim_{x \to a}(f(x).h(x)) = \begin{cases} & H > 0 \\ & H = 0 \\ & H < 0 \end{cases}$$

8.

$$\lim_{x \to a} \frac{h(x)}{f(x)} =$$

Theorem

5.

$$\lim_{x\to a} cf(x) = \begin{cases} +\infty & c > 0 \\ 0 & c = 0 \\ -\infty & c < 0 \end{cases}$$

6.

$$\lim(f(x).g(x)) = +\infty.$$

7.

$$\lim_{x\to a}(f(x).h(x)) = \begin{cases} +\infty & H > 0 \\ undetermined & H = 0 \\ -\infty & H < 0 \end{cases}$$

8.

$$\lim_{x\to a} \frac{h(x)}{f(x)} = 0.$$

Example

Consider the following three functions:

$$f(x) = x^{-2} \qquad g(x) = 2x^{-2} \qquad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x\to 0} f(x) = +\infty \qquad \lim_{x\to 0} g(x) = +\infty \qquad \lim_{x\to 0} h(x) = +\infty.$$

Then

1. $$\lim_{x\to 0}(f(x) - g(x)) =$$
2. $$\lim_{x\to 0}(f(x) - h(x)) =$$
3. $$\lim_{x\to 0}(g(x) - h(x)) =$$

Example

Consider the following three functions:

$$f(x) = x^{-2} \qquad g(x) = 2x^{-2} \qquad h(x) = x^{-2} - 1.$$

Then

$$\lim_{x\to 0} f(x) = +\infty \qquad \lim_{x\to 0} g(x) = +\infty \qquad \lim_{x\to 0} h(x) = +\infty.$$

Then

1. $$\lim_{x\to 0}(f(x) - g(x)) = \lim_{x\to 0} x^{-2} = \infty$$

2. $$\lim_{x\to 0}(f(x) - h(x)) = \lim_{x\to 0}(1) = 1$$

3. $$\lim_{x\to 0}(g(x) - h(x)) = \lim_{x\to 0} x^{-2} + 1 = \infty$$

Outline For the Session Five

- Limit at Infinity
- Continuity
- Continuous from the left and from the right
- Arithmetic of continuity
- continuity of composites
- Intermediate Value Theorem

Example

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$\frac{1}{x^2}$	100	10000	10^6		10^6	10000	100

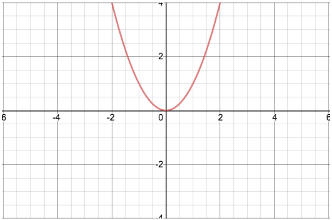
Consider that if

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a} g(x) = \infty$$

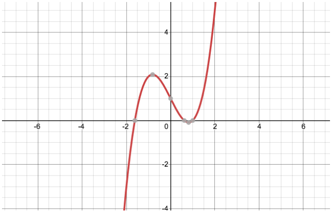
Then

$$\lim_{x\to a} (f(x) - g(x)) = \text{undetermined}$$

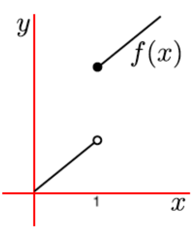
Continuity



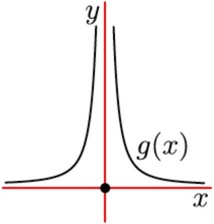
-4
-2
0
2
4
2
-2



-6
-4
-2
0
2
4
6
4
2
-2
-4

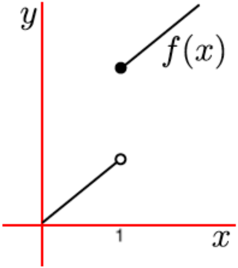


y
f(x)
1
x



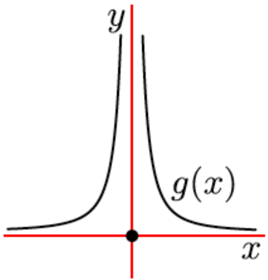
y
g(x)
x

$$f(x) = \begin{cases} x & x < 1 \\ x+2 & x \geq 1 \end{cases}$$



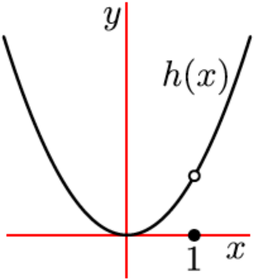
jump discontinuity

$$g(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



infinite discontinuity

$$h(x) = \begin{cases} \frac{x^3 - x^2}{x - 1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$



removable discontinuity

Outline - September 16, 2019

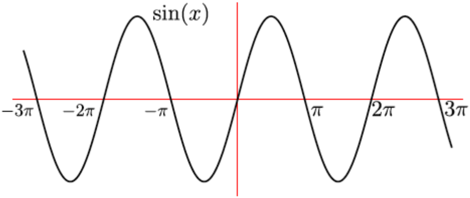
- **Section 1.6:**
  - Arithmetic of continuity
  - Continuity of composites
  - Intermediate Value Theorem
- **Section 2.1:**
  - Revisiting tangent lines

Arithmetic of continuity

Theorem

(**Arithmetic of continuity**) *Let $a, c \in \mathbb{R}$ and let $f(x)$ and $g(x)$ be functions that are continuous at a. Then the following functions are also continuous at $x = a$.*

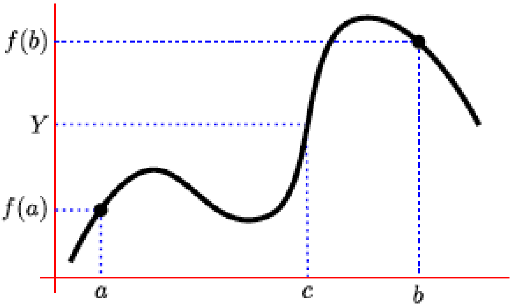
- $f(x) + g(x)$ *and* $f(x) - g(x)$,
- $cf(x)$ *and* $f(x)g(x)$, *and*
- $\frac{f(x)}{g(x)}$ *provided* $g(a) \neq 0$.


sin(x)
−3π
−2π
−π
π
2π
3π

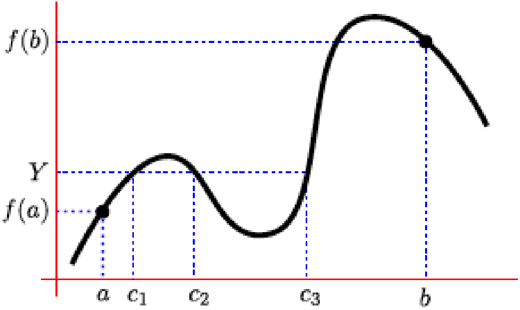
Intermediate value theorem(IVT)

Theorem

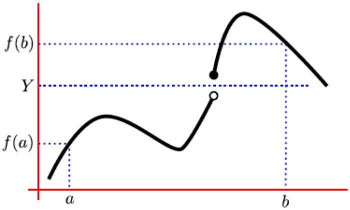
(Intermediate value theorem(IVT))

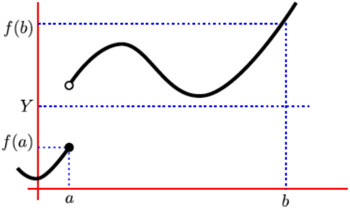


The existence not the uniqueness of c in IVT



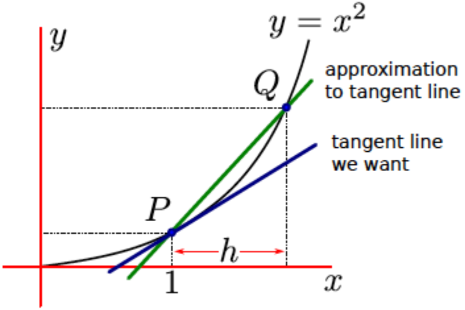
Not continuous functions at $[a, b]$ do not satisfy IVT





Revisiting tangent lines

Revisiting tangent lines



$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \leftarrow \quad \text{slope of the tangent line at } x = 1$$

Definition of the derivative

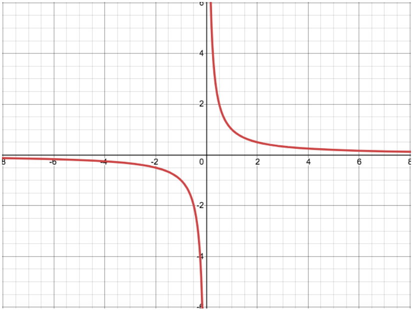
- $$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

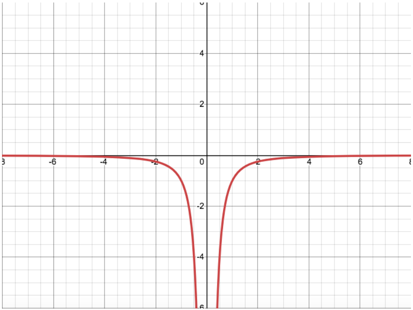
Examples

- $f(x) = c$
- $f(x) = x$
- $f(x) = x^2$
- $f(x) = \frac{1}{x}$
- $f(x) = \sqrt{x}$
- $f(x) = |x|$

$y = \frac{1}{x}$ and its derivative $-\frac{1}{x^2}$

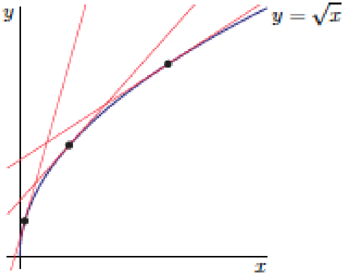


$$y = \frac{1}{x}$$

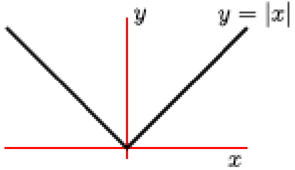


$$y = \frac{-1}{x^2}$$

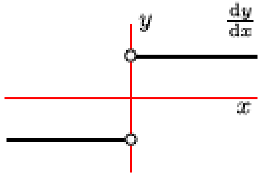
Tangent lines to $y = \sqrt{x}$



The derivative of the function $f(x) = |x|$: not differentiable at $x = 0$

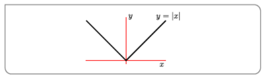


The derivative of the function $f(x) = |x|$

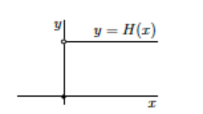


Where a function is not differentiable at $x = a$?

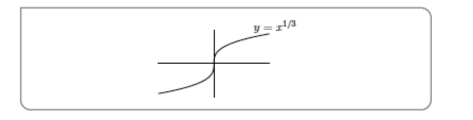
- Having a Sharp Corner at $x = a$



- The function is not continuous at $x = a$



- Having a tangent line, but the slope of the tangent line at $x = a$ is infinity

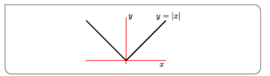


Outline - September 20, 2019

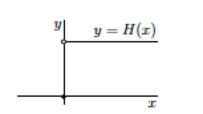
- **Section 2.2:**
  - Not differentiable examples
  - The relation between continuous and differentiable functions
- **Section 2.3:**
  - Interpretations of the derivative

Where a function is not differentiable at $x = a$?

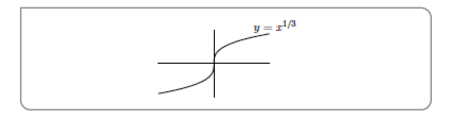
- Having a Sharp Corner at $x = a$



- The function is not continuous at $x = a$

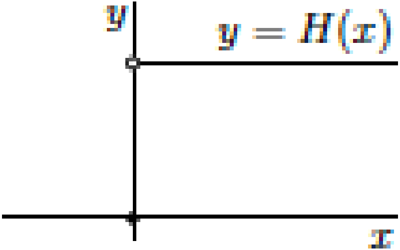


- Having a tangent line, but the slope of the tangent line at $x = a$ is infinity



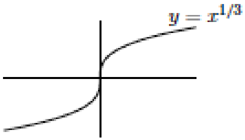
An example of a discontinuous and not differentiable function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$



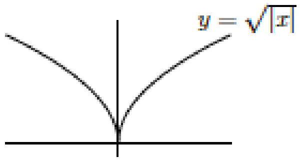
An example of a function with a tangent line with slope infinity at $x = 0$

$f(x) = x^{1/3}$

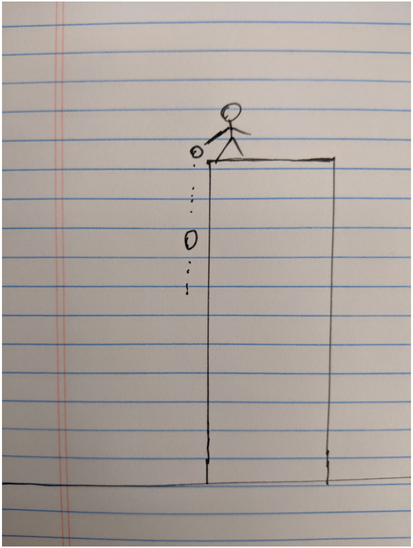


An example of a continuous and **not** differentiable function

$y = \sqrt{|x|}$



Instantaneous rate of change



average rate of change of $f(t)$ from $t = a$ to $t = a + h$ is

$$\frac{\text{change in } f(t) \text{ from } t = a \text{ to } t = a + h}{\text{length of time from } t = a \text{ to } t = a + h}$$

$$= \frac{f(a+h) - f(a)}{h}.$$

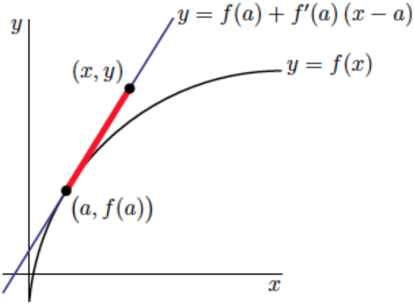
And so

instantaneous rate of change of $f(t)$ at $t = a$

$$= \lim_{h \to 0} [\text{average rate of change of } f(t) \text{ from } t = a \text{ to } t = a + h]$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

Finding tangent line to a curve at $x = a$



A line segment of a tangent line

$$y = f(a) + f'(a)(x - a)$$

Outline - September 23, 2019

- **Section 2.4 and 2.5:**
  - Derivative of some simple functions
  - Tools
  - Examples

A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0 \qquad \frac{d}{dx}x = 1 \qquad \frac{d}{dx}x^2 = 2x \qquad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

A list of derivative of some simple functions:

$$\frac{d}{dx}1 = 0 \qquad \frac{d}{dx}x = 1 \qquad \frac{d}{dx}x^2 = 2x \qquad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Tools

Let $f(x)$ and $g(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- $\frac{d}{dx}\{f(x) + g(x)\} = f'(x) + g'(x)$
- $\frac{d}{dx}\{f(x) - g(x)\} = f'(x) - g'(x)$
- $\frac{d}{dx}\{cf(x)\} = cf'(x)$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- $\frac{d}{dx}\{\frac{f(x)}{g(x)}\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \qquad g(x) \neq 0$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- $\frac{d}{dx}\{\frac{f(x)}{g(x)}\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \qquad g(x) \neq 0$
- $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- $\frac{d}{dx}\{\frac{1}{g(x)}\} = \frac{-g'(x)}{g(x)^2} \qquad g(x) \neq 0$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

- $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- $\frac{d}{dx}\{\frac{f(x)}{g(x)}\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \qquad g(x) \neq 0$
- $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- $\frac{d}{dx}\{\frac{1}{g(x)}\} = \frac{-g'(x)}{g(x)^2} \qquad\qquad g(x) \neq 0$
- $\frac{d}{dx}\{f(x)g(x)h(x)\} =$ $f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- $\frac{d}{dx}\{f(x)^n\} = nf^{n-1}(x)f'(x)$

Tools

Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions and let $c, d \in \mathbb{R}$.

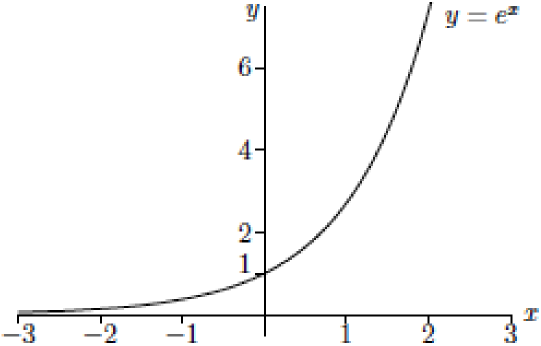
- $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + g'(x)f(x)$
- $\frac{d}{dx}\{\frac{f(x)}{g(x)}\} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \qquad g(x) \neq 0$
- $\frac{d}{dx}\{cf(x) + dg(x)\} = cf'(x) + dg'(x)$
- $\frac{d}{dx}\{f(x)^2\} = 2f(x)f'(x)$
- $\frac{d}{dx}\{\frac{1}{g(x)}\} = \frac{-g'(x)}{g(x)^2} \qquad g(x) \neq 0$
- $\frac{d}{dx}\{f(x)g(x)h(x)\} =$ $f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- $\frac{d}{dx}\{f(x)^n\} = nf^{n-1}(x)f'(x)$
- Let a be a rational number, then

$$\frac{d}{dx}x^a = ax^{a-1}.$$

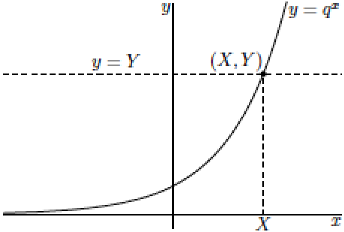
Outline - September 25, 2019

- **Section 2.7 and 2.8:**
  - Derivative of exponential functions
  - Derivative of trigonometric functions

The graph of e^x



The graph of q^x where $q > 1$



YOUR TURN!

Example

Find a such that the following function is continuous.

$$f(x) = \begin{cases} e^{x+a} & x < 0 \\ \sqrt{x+1} & x \geq 0 \end{cases}$$

Example

We have

1. $\log_q(xy) =$
   (a) $\log_q(x) + \log_q(y)$
   (b) $\log_q(x)\log_q(y)$
2. $\log_q(x/y) =$
3. $\log_q(x^r) =$

Example

We have

1. $\log_q(xy) = \log_q(x) + \log_q(y)$.
   The reason for this is that

$$q^{\log_q(xy)} = xy = q^{\log_q(x)} q^{\log_q(y)} = q^{\log_q(x)+\log_q(y)}$$

   Therefore, $\log_q(xy) = \log(x) + \log(y)$.
2. $\log_q(x/y) = \log_q(x) - \log_q(y)$
3. $\log_q(x^r) = r \log_q(x)$

TOOLS:

$$\frac{d}{dx}(f \circ g)(x) = g'(x)f'(g(x))$$

A list of derivative of some simple functions:

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = (\log_e a)a^x$$

Example

Find the derivative of $2^{\sqrt{x}}$.

Example

Find the derivative of $2^{\sqrt{x}}$.

Example

Find a and b such that the following function is differentiable.

$$f(x) = \begin{cases} x^3 + a & x < 1 \\ e^{x-1} + bx & x \geq 1 \end{cases}$$

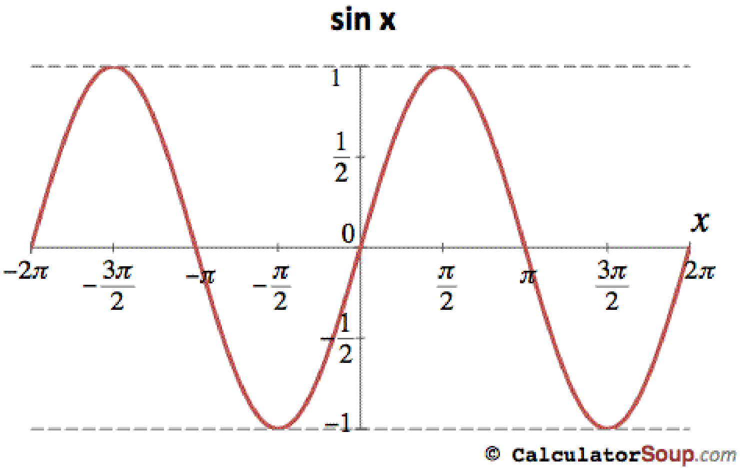
Outline - September 30, 2019

- **Section 2.8, 2.9, 0.6:**
  - Derivative of trigonometric functions
  - The chain rule
  - inverse of a function

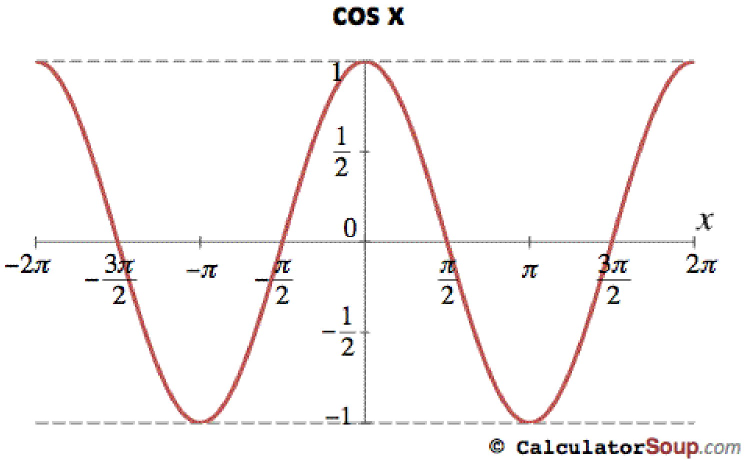
A list of derivative of some simple functions:

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = (\log_e a)a^x$$

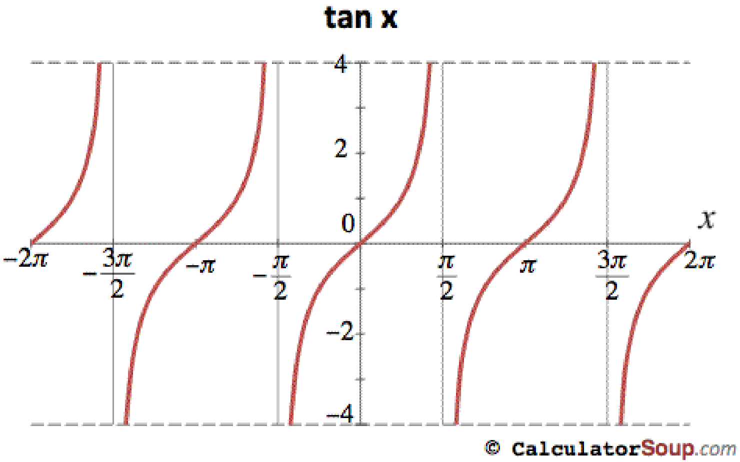
$\sin(x)$ domain $= \mathbb{R}$ range $= [-1, 1]$



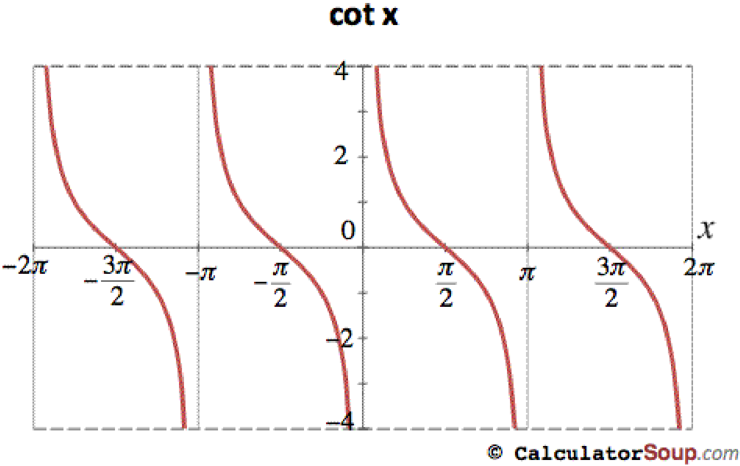
$\cos(x)$ domain $= \mathbb{R}$ range $= [-1, 1]$



$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \text{domain} = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R}$$

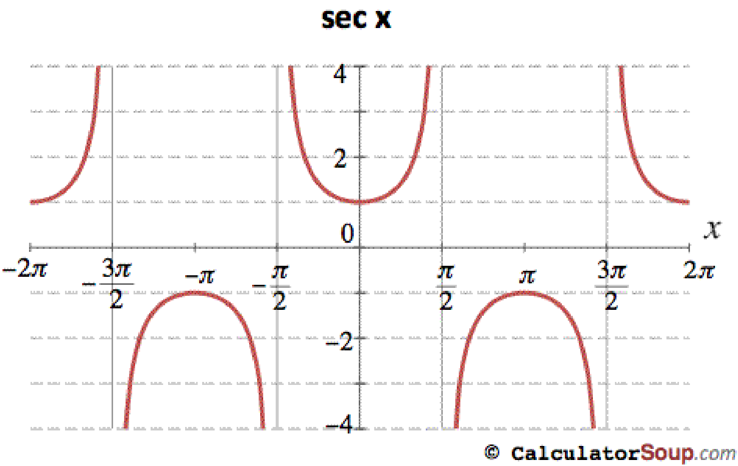


$$\cot(x) = \frac{\cos(x)}{\sin(x)} \quad \text{domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R}$$

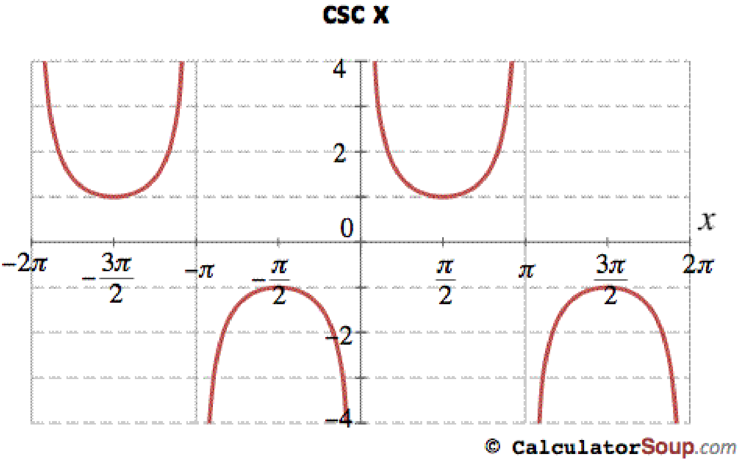


$$\sec(x) = \frac{1}{\cos(x)} \quad \text{domain} = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$$

$$\text{range} = \mathbb{R} - (-1, 1)$$



$$\csc(x) = \frac{1}{\sin(x)} \quad \text{domain} = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\} \quad \text{range} = \mathbb{R} - (-1, 1)$$



Derivative of $\sin(x)$

Question: Knowing that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

compute the derivative of $\sin(x)$ at $x = 0$.

Derivative of $\sin(x)$

Question: Knowing that

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

compute the derivative of $\sin(x)$ at $x = 0$.

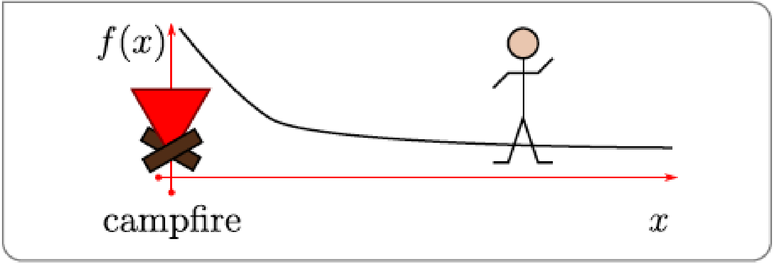
(**sandwich (or squeeze or pinch) theorem**) Let $a \in \mathbb{R}$ and let f, g, h be three functions so that $f(x) \leq g(x) \leq h(x)$ for all x in an interval around a, except possibly at $x = a$. Then if

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then it is also the case that

$$\lim_{x \to a} g(x) = L.$$

An example of the application of the chain rule



- Your position at time t is $x(t)$.
- The temperature of the air at position x is $f(x)$.
- The temperature that you feel at time t is $F(t) = f(x(t))$.
- The instantaneous rate of change of temperature that you feel is $F'(t)$.

The chain rule

Theorem

Let f and g be differentiable functions then

$$\frac{d}{dx}f(g(x)) = f'(g(x)).g'(x)$$

The chain rule

Theorem

Let $y = f(u)$ and $u = g(x)$ be differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

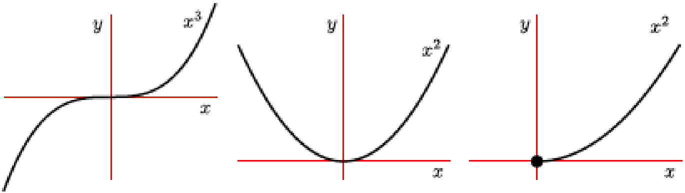
Outline - October 2, 2019

- **Section 0.6, 2.10:**
  - Inverse of a function
  - Natural logarithm

input number $x \mapsto f$ does "stuff" to $x \mapsto$ return number y

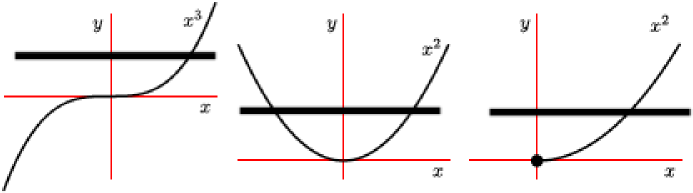
take output $y \mapsto$ do "stuff" to $y \mapsto$ return the original number x

One-to-one functions



$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array} \qquad \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array} \qquad \begin{array}{ccc} [0, \infty] & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

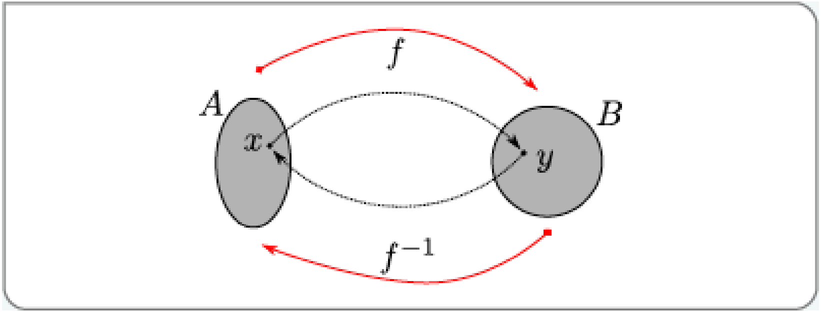
One-to-one functions



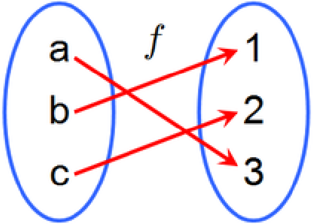
$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array} \qquad \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array} \qquad \begin{array}{ccc} [0,\infty) & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

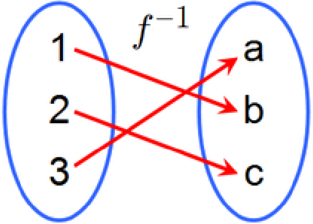
is one-to-one   is not one-to-one   is one-to-one

Inverse of a functions

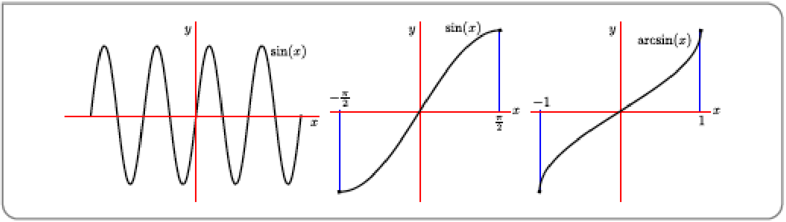


Inverse of a functions

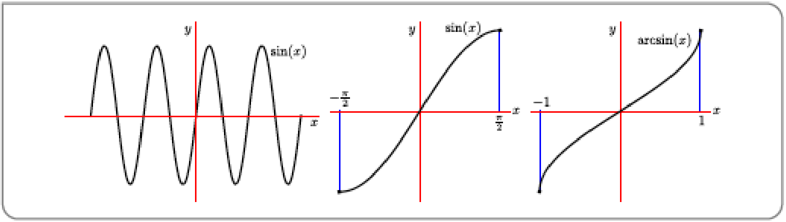




Inverse of $\sin(x)$

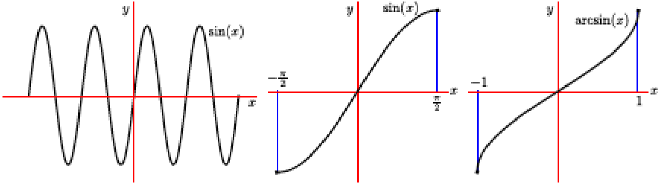


Inverse of $\sin(x)$



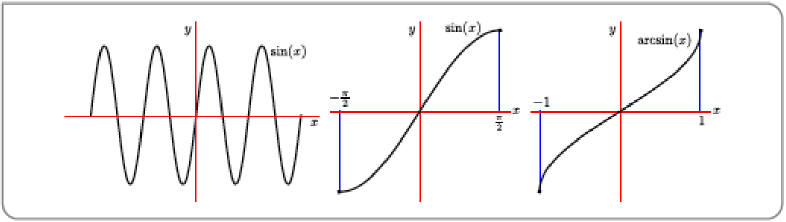
- $\sin(x)$ is not invertible on the domain $\mathbb{R}$ because it is not one-to-one.

Inverse of $\sin(x)$



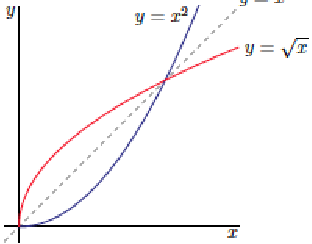
- $\sin(x)$ is not invertible on the domain $\mathbb{R}$ because it is not one-to-one.
- If we look at $\sin(x)$ on the domain $[-\pi/2, \pi/2]$, then it is one-to-one, and so it is has an inverse.

Inverse of $\sin(x)$



- $\sin(x)$ is not invertible on the domain $\mathbb{R}$ because it is not one-to-one.
- If we look at $\sin(x)$ on the domain $[-\pi/2, \pi/2]$, then it is one-to-one, and so it is has an inverse.
- The inverse of $\sin(x)$ is $\arcsin(x)$ on the domain $[-1, 1]$ and with the range $[-\pi/2, \pi/2]$.

How to find the inverse of a function by its graph



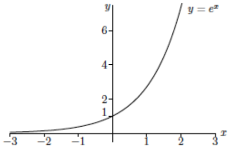
$a^{\log_a x} = x$

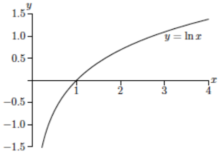
Remember that for $a > 1$,

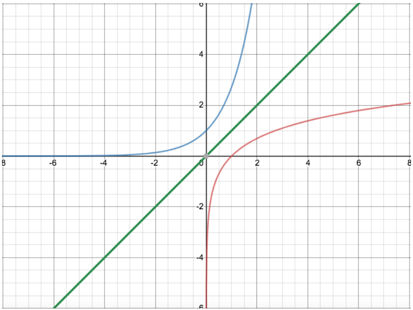
$$a^{\log_a x} = x,$$

$$\log_a x = \frac{\log_e x}{\log_e a}.$$

The inverse of e^x







Outline - October 4, 2019

- **Section 2.10 and 2.11:**
  - Natural logarithm
  - Implicit derivative

Useful facts!

- $\frac{d}{dx}a^x = (\ln a)a^x$.
- $\log_a x = \frac{\ln x}{\ln a}$ $\qquad \ln x = \frac{\log_a x}{\log_a e}$ $\qquad a > 1$.
- $\ln(xy) = \ln x + \ln y$.
- $\ln(x/y) = \ln x - \ln y$.
- $\ln x^r = r \ln x$.

Useful facts!

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- $\ln(xy) = \ln x + \ln y$.
- $\ln(x/y) = \ln x - \ln y$.
- $\ln x^r = r \ln x$.
- $\frac{d}{dx} \ln x = \frac{1}{x}$.

Useful facts!

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- $\log_a x = \frac{\ln x}{\ln a} \qquad \ln x = \frac{\log_a x}{\log_a e} \qquad a > 1.$
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- $\ln(x/y) = \ln x - \ln y$.
- $\ln x^r = r \ln x$.
- $\frac{d}{dx} \ln x = \frac{1}{x}$.
- $\frac{d}{dx} \ln |x| = \frac{1}{x}$.

Useful facts!

- $\frac{d}{dx} a^x = (\ln a) a^x$.
- $\log_a x = \frac{\ln x}{\ln a}$ $\qquad \ln x = \frac{\log_a x}{\log_a e}$ $\qquad a > 1$.
- $\ln(xy) = \ln x + \ln y$.
- $\ln(x/y) = \ln x - \ln y$.
- $\ln x^r = r \ln x$.
- $\frac{d}{dx} \ln x = \frac{1}{x}$.
- $\frac{d}{dx} \ln |x| = \frac{1}{x}$.
- $\frac{d}{dx} \log_a x = \frac{1}{x . \ln a}$.

Useful facts!

- $\frac{d}{dx}a^x = (\ln a)a^x$.
- $\log_a x = \frac{\ln x}{\ln a}$ $\qquad \ln x = \frac{\log_a x}{\log_a e}$ $\qquad a > 1$.
- $\ln(xy) = \ln x + \ln y$.
- $\ln(x/y) = \ln x - \ln y$.
- $\ln x^r = r \ln x$.
- $\frac{d}{dx}\ln x = \frac{1}{x}$.
- $\frac{d}{dx}\ln |x| = \frac{1}{x}$.
- $\frac{d}{dx}\log_a x = \frac{1}{x.\ln a}$.
- $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

Useful facts!

- $\frac{d}{dx} a^x = (\ln a) a^x$.
- $\log_a x = \frac{\ln x}{\ln a}$ $\qquad$ $\ln x = \frac{\log_a x}{\log_a e}$ $\qquad$ $a > 1$.
- $\ln(xy) = \ln x + \ln y$.
- $\ln(x/y) = \ln x - \ln y$.
- $\ln x^r = r \ln x$.
- $\frac{d}{dx} \ln x = \frac{1}{x}$.
- $\frac{d}{dx} \ln |x| = \frac{1}{x}$.
- $\frac{d}{dx} \log_a x = \frac{1}{x . \ln a}$.
- $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$
- $\frac{d}{dx} |f(x)| = \frac{f'(x)}{f(x)}$.

Outline - October 7, 2019

- **Section 2.11 and 2.12:**
  - Implicit derivative
  - Derivative of Trig functions

Implicit derivative

$$\frac{d}{dx}x = \frac{d}{dx}e^{\ln x} \qquad (\frac{d}{dx}x = \frac{d}{dx}e^{y})$$

which is the same as

$$1 = (\frac{d}{dx}\ln x).e^{\ln x} \qquad (1 = y'e^{y}).$$

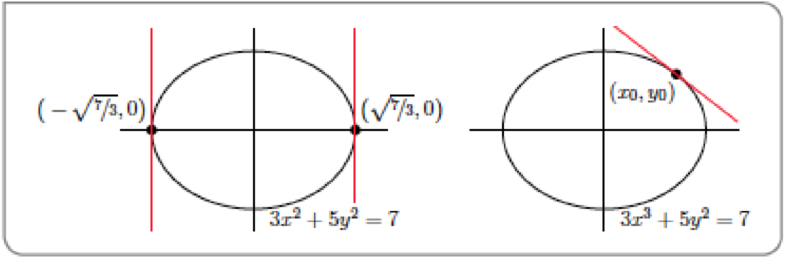
Note that $e^{\ln x} = x (e^{y} = x)$, thus

$$1 = (\frac{d}{dx}\ln x).x \qquad (1 = y'x)$$

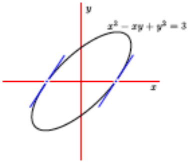
and so

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad (y' = \frac{1}{x}).$$

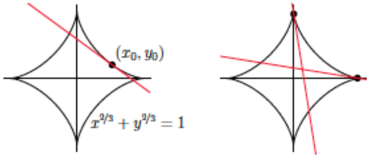
$3x^3 + 5y^2 = 7$



$x^2 - xy + y^2 = 3$



$x^{2/3} + y^{2/3} = 1$



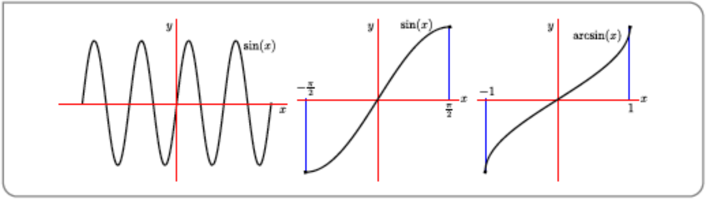
Outline - October 9, 2019

- **Section 2.12:**
  - Derivative of Trig functions

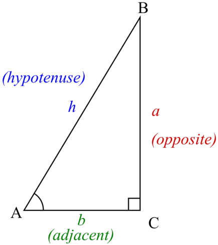
Review of the inverse of a function

Remember that the inverse of a one-to-one function $f(x)$ with domain A and range B is a function $g(x)$ with domain B and range A such that

$$f(g(y)) = y \qquad g(f(x)) = x \qquad x \in A, y \in B.$$



Trigonometry



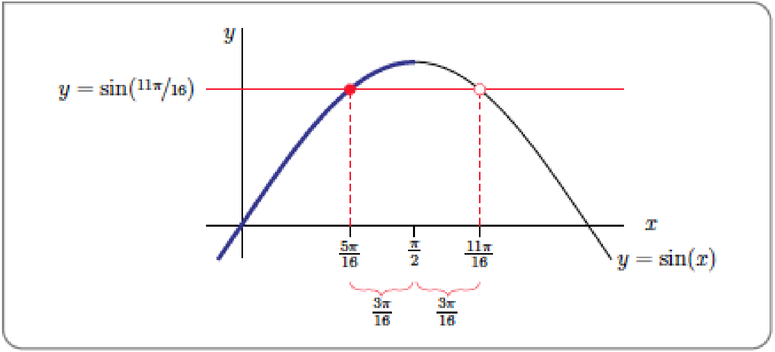
- **sine:** $\sin A = \frac{a}{h} = \frac{\text{opposite}}{\text{hypotenuse}}$
- **cosine:** $\cos A = \frac{b}{h} = \frac{\text{adjacent}}{\text{hypotenuse}}$
- **tangent:** $\tan A = \frac{a}{b} = \frac{\text{opposite}}{\text{adjacent}}$
- **cosecant:** $\csc A = \frac{h}{a} = \frac{\text{hypotenuse}}{\text{opposite}}$
- **secant:** $\sec A = \frac{h}{b} = \frac{\text{hypotenuse}}{\text{adjacent}}$
- **cotangent:** $\cot A = \frac{b}{a} = \frac{\text{adjacent}}{\text{opposite}}$

arcsin(sin(x))

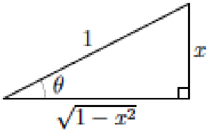
$\arcsin(\sin(x)) =$ the unique angle θ between $-\pi/2$ and $\pi/2$ obeying that

$$\sin(x) = \sin(\theta).$$

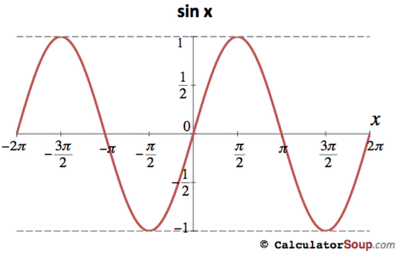
What is $\arcsin(\sin(\frac{11\pi}{16}))$?

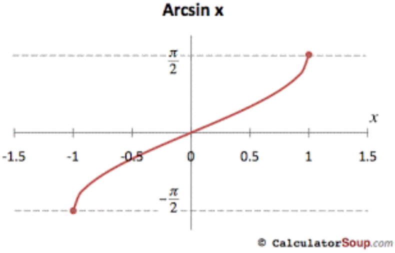


$\cos(\arcsin(x)) = \sqrt{1 - x^2}$

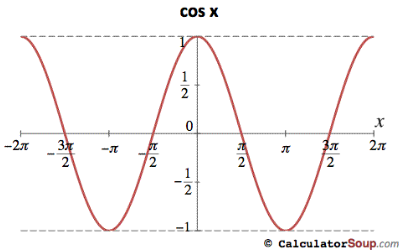


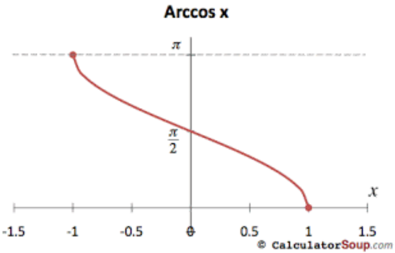
Inverse of $\sin(x)$



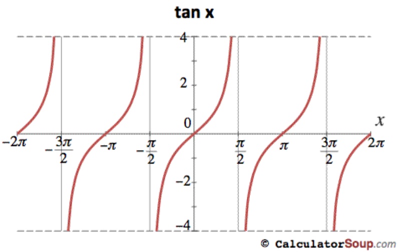


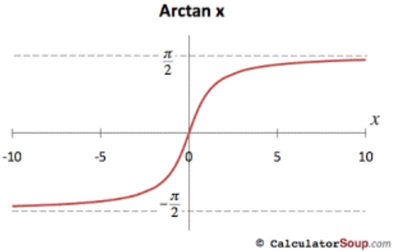
Inverse of cos(x)



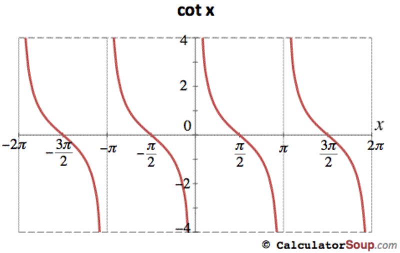


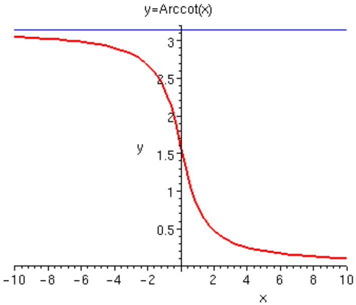
Inverse of tan(x)





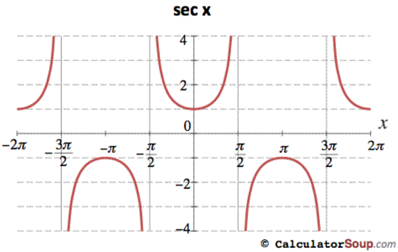
Inverse of $\cot(x)$

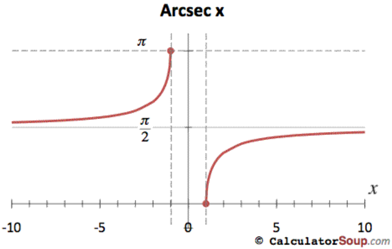




Inverse of $\sec(x)$

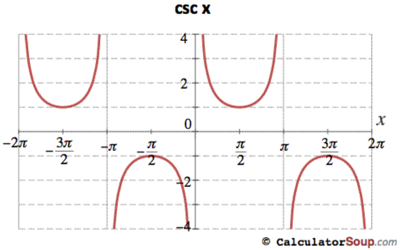
$$\mathrm{arcsec}(x) = \arccos(1/x)$$

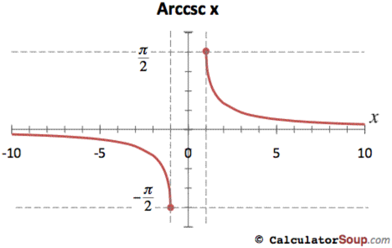




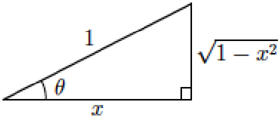
Inverse of $\csc(x)$

$$\mathrm{arccsc}(x) = \arcsin(1/x)$$

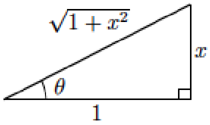




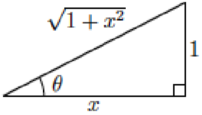
$$\sin(\theta) = \sin(\arccos(x)) = \sqrt{1 - x^2}$$



$$\cos^2(\arctan(x)) = \cos^2(\theta) = \frac{1}{1+x^2}.$$



$$\frac{1}{\csc^2(\theta)} = \sin^2(\theta) = 1 + x^2$$

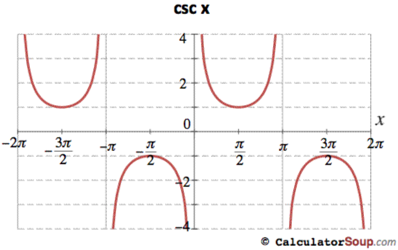


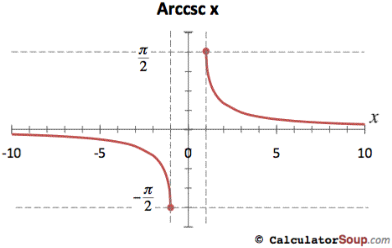
Outline - October 11, 2019

- **Section 3.1:**
  - Derivative of Trig functions

Inverse of csc(x)

$$\operatorname{arccsc}(x) = \arcsin(1/x)$$





Derivative of the inverses of trigonometric functions in a nutshell

In a nutshell the derivatives of the inverse trigonometric functions are

$$\frac{d}{dx}\,\text{arcsin}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\operatorname{arccsc}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\,\text{arccos}(x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\,\text{arctan}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arccot}(x) = -\frac{1}{1+x^2}$$

The Application of Derivatives

Velocity and Acceleration

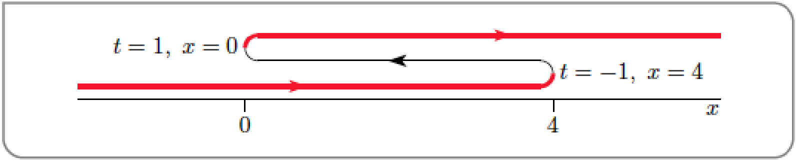
If you are moving along the x-axis and your position at time t is $x(t)$, then

- your velocity at time t is $v(t) = x'(t)$ and
- your acceleration at time t is $a(t) = v'(t) = x''(t)$.

Direction of your move with $x(t) = t^3 - 3t + 2$

t	$(t-1)(t+1)$	$x'(t) = 3(t-1)(t+1)$	Direction
$t < -1$	*positive*	*positive*	*right*
$t = -1$	*zero*	*zero*	*halt*
$-1 < t < 1$	*negative*	*negative*	*left*
$t = 1$	*zero*	*zero*	*halt*
$t > 1$	*positive*	*positive*	*right*

And here is a schematic picture of the whole trajectory.



Direction of your move with $x(t) = t^3 - 12t + 5$

t	$(t-2)(t+2)$	$x'(t) = 3(t-2)(t+2)$	Direction
$t < -2$	*positive*	*positive*	*right*
$t = -2$	*zero*	*zero*	*halt*
$-2 < t < 2$	*negative*	*negative*	*left*
$t = 2$	*zero*	*zero*	*halt*
$t > 2$	*positive*	*positive*	*right*

t	*your position* $x(t)$	$x'(t)$	Direction
0	5	*negative*	*left*
$t = 2$	-11	*zero*	*halt*
$t = 10$	885	*positive*	*right*

Outline - October 16, 2019

- **Section 3.2: Exponential Growth and Decay**
  - 3.1: Carbon Dating

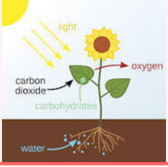
EXAM: Friday, October 18, Here in Class, at 2pm

Carbon Dating

Cosmic ray hitting atmosphere

Nitrogen $(N) \rightarrow$ *Carbon*(C)

Vegetation absorbs C *through photosynth*



Animals acquire C *by eating plants*

C *decreases when animal dies*

More precisely, let $Q(t)$ denote the amount of C (an element) in the plant or animal t years after it dies. The number of radioactive decays (rate of change) per unit time, at time t, is proportional to the amount of C present at time t, which is $Q(t)$. Thus

Radioactive Decay

$$\frac{dQ}{dt}(t) = -kQ(t) \tag{1}$$

Corollary

The function $Q(t)$ satisfies the equation

$$\frac{dQ}{dt} = -kQ(t)$$

if and only if

$$Q(t) = Q(0).e^{-kt}$$

The half-life (the half-life of C is the length of time that it takes for half of the C to decay) is defined to be the time $t_{1/2}$ which obeys

$$Q(t_{1/2}) = \frac{1}{2}.Q(0).$$

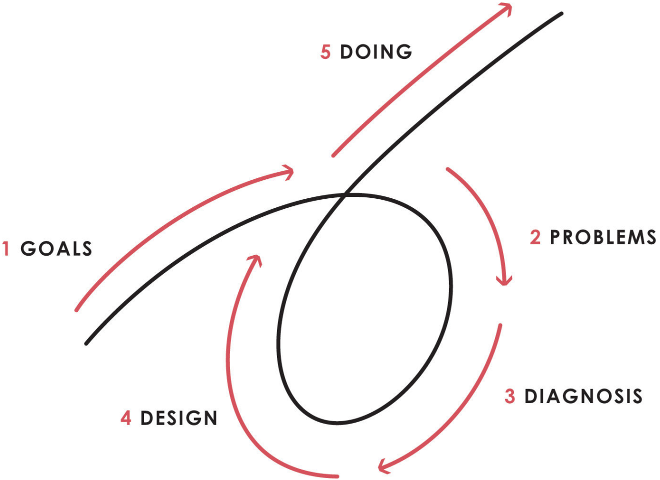
The half-life is related to the constant k by

$$t_{1/2} = \frac{\ln 2}{k}.$$

Outline - October 21, 2019

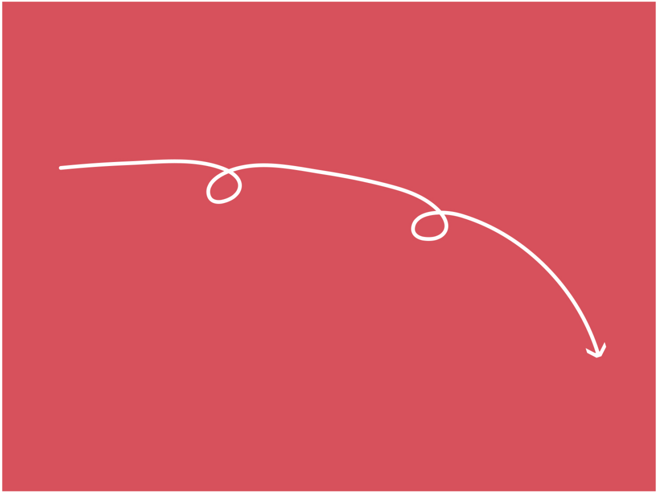
- **Section 3.3.2: Newton's Law of Cooling**
  - 3.1: Newton's Law of Cooling

No pain no gain

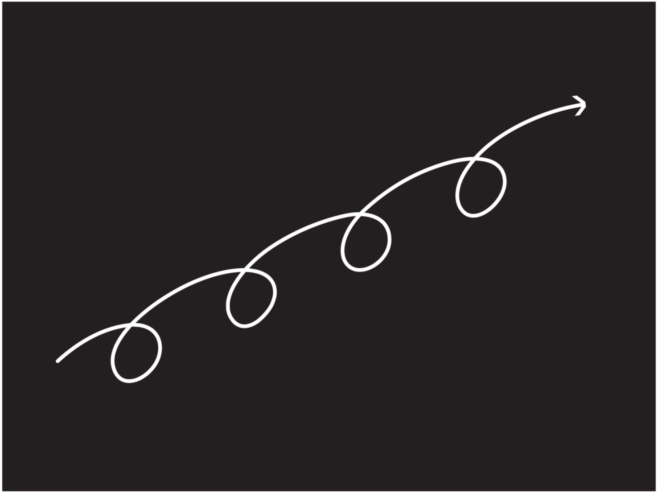


Principles (Ray Dalio)

Most people



Successful person



Newton's Law of Cooling

$$\frac{dT}{dt}(t) = K\,[T(t) - A]\,.$$

We have three possibilities:

- $T(t) > A \Rightarrow [T(t) - A] > 0$, thus the temperature of the body is decreasing, so $\frac{dT}{dt}$ must be negative, since $\frac{dT}{dt}(t) = K\,[T(t) - A]$, we must have $K < 0$.

- $T(t) < A \Rightarrow [T(t) - A] < 0$, thus the temperature of the body is increasing, so $\frac{dT}{dt}$ must be positive, since $\frac{dT}{dt}(t) = K\,[T(t) - A]$, we must have $K < 0$.

- $T(t) = A \Rightarrow [T(t) - A] = 0$, thus the temperature of the body is no changing, so $\frac{dT}{dt}$ must be zero, since $\frac{dT}{dt}(t) = K\,[T(t) - A]$. This does not impose any condition on K.

Newton's Law of Cooling

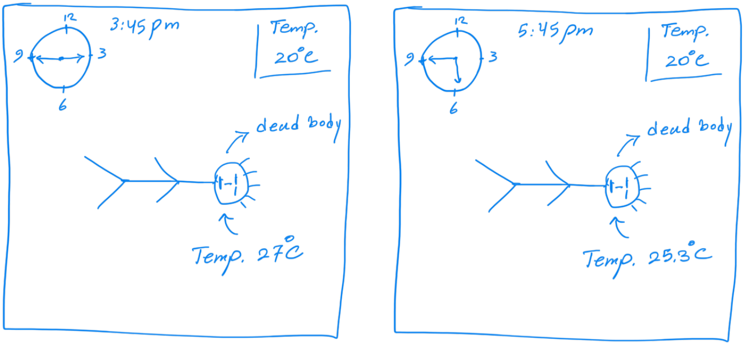
Corollary

A differentiable function $T(t)$ obeys the differential equation

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

if and only if

$$T(t) = [T(0) - A]e^{Kt} + A.$$



I entered this room at 9:30 am and talk to him for 5 minutes.

Outline - October 23, 2019

- **Section 3.3.3: Population Growth**
- **Section 3.2: Related Rates**

Population Growth

Suppose that we wish to predict the size $P(t)$ of a population as a function of the time t. So suppose that in average each couple produces β offspring (for some constant β) and then dies. Then over the course of one generation since we have $P(t)/2$ couples and each have produced β offspring, thus the population of the children of one generation is

$$\beta\frac{P(t)}{2}.$$

Let t_g be the life span of one generation, then

$$\begin{aligned} P(t+t_g) &= \beta\frac{P(t)}{2} \\ &= P(t) + \beta\frac{P(t)}{2} - P(t). \end{aligned}$$

Therefore,

$$P(t+t_g) - P(t) = \beta \frac{P(t)}{2} - P(t)$$

and so dividing both sides by t_g, we have

$$\begin{aligned}\frac{P(t+t_g) - P(t)}{t_g} &= \frac{1}{t_g}\left(\frac{\beta}{2}P(t) - P(t)\right) \\ &= \frac{1}{t_g}\left(\frac{\beta}{2} - 1\right)P(t)\end{aligned}$$

Let $\frac{1}{t_g}\left(\frac{\beta}{2} - 1\right) = b$, then

$$\frac{P(t+t_g) - P(t)}{t_g} = bP(t).$$

Approximately, we have

$$\frac{dP}{dt} = bP(t).$$

Moreover, same as the model for carbon dating we can write

$$P(t) = P(0)e^{bt}.$$

Malthusian growth model

Malthusian growth model

The model for the population growth is

$$\frac{dP}{dt} = bP(t)$$

and $P(t)$ satisfies the above equation if and only if

$$P(t) = P(0)e^{bt}.$$

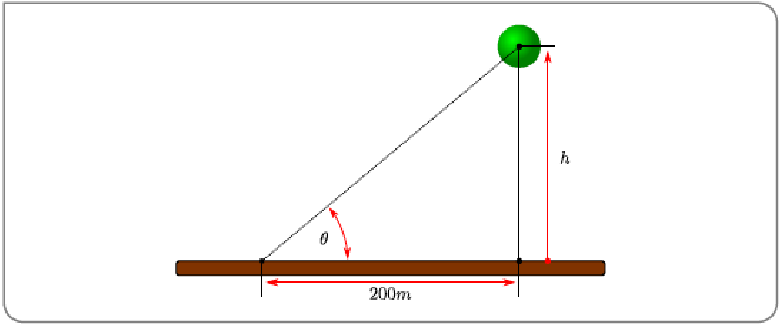
Related Rates

Volume of a sphere

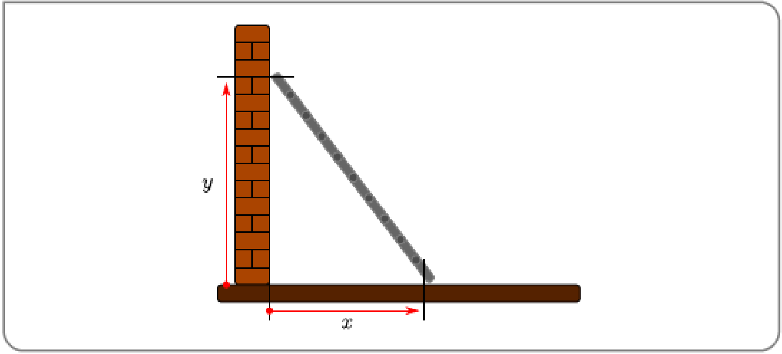
Remember that the volume of a sphere with radius r is

$$V = \frac{4}{3}\pi r^3.$$

Helium Balloon



Ladder

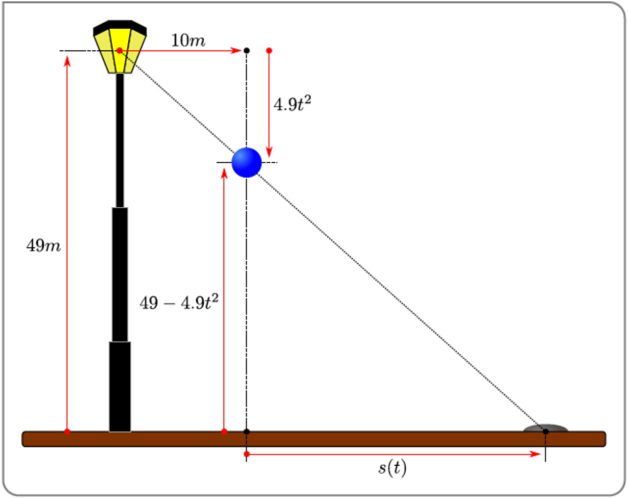


Outline - October 25, 2019

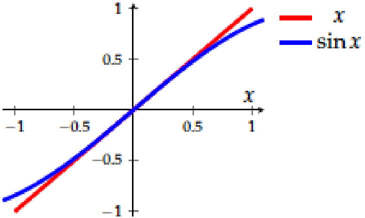
- **Section 3.2: Related Rates: An Example**
- **Section 3.4.2 The Linear Approximation**
- **Section 3.4.3 The Quadratic Approximation**

Shadow of the Ball

Similar triangles-ratio



Approximation

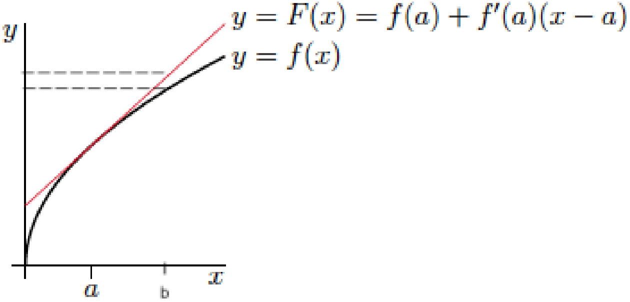


This figure shows that the curve $y = x$ and $y = \sin(x)$ are almost the same when x is close to 0. Hence if we want the value of $\sin(1/10)$ we just use this approximation $y = x$ to get

$$\sin(1/10) \approx 1/10.$$

The linear approximation

Given a function $f(x)$ we want to have the approximating function to be a linear function that is $F(x) = A + Bx$ for some constants A and B.



The linear approximation

$$f(x) \approx F(x) = f(a) + f'(a)(x - a)$$

Example

Estimate $e^{0.01}$? So $f(x) = e^x$ and $a = 0$.

The quadratic approximation

In linear approximation we had

$$f(x) \approx F(x) = f(a) + f'(a)(x-a) \Rightarrow$$

$$f(a) = F(a) \quad \text{and} \quad f'(a) = F'(a).$$

We now want our approximation function to be a quadratic function of x, that is, $F(x) = A + Bx + Cx^2$. To have a good approximating function we choose A, B, and C so that

- $f(a) = F(a)$
- $f'(a) = F'(a)$
- $f''(a) = F''(a)$

These conditions give us the following equations

$$F(x) = A + Bx + Cx^2 \quad \Rightarrow \quad F(a) = A + Ba + Ca^2 = f(a)$$

$$F'(x) = B + 2Cx \quad \Rightarrow \quad F'(a) = B + 2Ca = f'(a)$$

$$F''(x) = 2C \quad \Rightarrow \quad F'(a) = 2C = f''(a)$$

Solving these equation we can write A, B, and C in terms of $f(a)$, $f'(a)$, and $f''(a)$. So that

$$C = \frac{1}{2} f''(a)$$

$$B = f'(a) - af''(a)$$

$$A = f(a) - a[f'(a) - af''(a)] - \frac{1}{2} f''(a)a^2.$$

Consider that $F(x) = A + Bx + CX^2$, substituting A, B, and C, we obtain

Quadratic Approximation

$$F(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

Therefore,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

Outline - October 28, 2019

- **Section 3.4.3 The Quadratic Approximation**
- **Section 3.4.4 Taylor Polynomials**
- **Section 3.4.5 Some Examples**

Linear Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x - a)$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$

Linear Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x - a)$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$

Then

$$F(a) = c_0 = f(a) \qquad\qquad F'(a) = c_1 = f'(a).$$

And so

$$F(x) = f(a) + f'(a)(x - a).$$

Quadratic Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x - a) + c_2(x - a)^2$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$
3. $F''(a) = f''(a)$

Quadratic Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x-a) + c_2(x-a)^2$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$
3. $F''(a) = f''(a)$

Then

$$F(a) = c_0 = f(a) \qquad F'(a) = c_1 = f'(a) \qquad F''(a) = 2c_2 = f''(a).$$

Quadratic Approximation

Approximate $f(x)$ by $F(x) = c_0 + c_1(x-a) + c_2(x-a)^2$ such that

1. $F(a) = f(a)$
2. $F'(a) = f'(a)$
3. $F''(a) = f''(a)$

Then

$$F(a) = c_0 = f(a) \qquad F'(a) = c_1 = f'(a) \qquad F''(a) = 2c_2 = f''(a).$$

And so

$$F(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

Taylor Polynomial

We want to approximate $f(x)$ with a polynomial $T_n(x)$ of degree n of the form

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n$$

such that

1. $T_n(a) = f(a)$,
2. $T_n'(a) = f'(a)$,

   $\vdots$

n. $T_n^{(n)}(a) = f^{(n)}(a)$.

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) =$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) =$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) = c_1 = f'(a)$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) = c_1 = f'(a)$$

$$T_n''(x) = 2c_2 + 3 \times 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n''(a) =$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) = c_1 = f'(a)$$

$$T_n''(x) = 2c_2 + 3 \times 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n''(a) = 2c_2 = f''(a)$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) = c_1 = f'(a)$$

$$T_n''(x) = 2c_2 + 3 \times 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n''(a) = 2c_2 = f''(a)$$

$$T_n^{(3)}(x) = 3 \times 2c_3 + 4 \times 3 \times 2c_4(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n^{(3)}(a) =$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) = c_1 = f'(a)$$

$$T_n''(x) = 2c_2 + 3 \times 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n''(a) = 2c_2 = f''(a)$$

$$T_n^{(3)}(x) = 3 \times 2c_3 + 4 \times 3 \times 2c_4(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n^{(3)}(a) = 6c_3 = f^{(3)}(a)$$

Taylor Polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n \Rightarrow T_n(a) = c_0 = f(a)$$

$$T_n'(x) = c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} \Rightarrow T_n'(a) = c_1 = f'(a)$$

$$T_n''(x) = 2c_2 + 3 \times 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n''(a) = 2c_2 = f''(a)$$

$$T_n^{(3)}(x) = 3 \times 2c_3 + 4 \times 3 \times 2c_4(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$
$$\Rightarrow T_n^{(3)}(a) = 6c_3 = f^{(3)}(a)$$

$$\vdots$$

$$T_n^{(n)}(x) = n!c_n \Rightarrow T_n^{(n)}(a) = n!c_n$$

Taylor Polynomial

We have

$$c_0 = f(a), c_1 = f'(a), c_2 = \frac{1}{2!}f''(a), c_3 = \frac{1}{3!}f^{(3)}(a), \ldots, c_n = \frac{1}{n!}f^{(n)}(a)$$

and

$$T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

we have that

$$f(x) \approx T_n(x) =$$

$$f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a) +$$

$$\frac{1}{3!}f^{(3)}(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Taylor Polynomial

Taylor Polynomial

Let a be a constant and let n be a non-negative integer. The nth degree Taylor polynomial for $f(x)$ about $x = a$ is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2$$

$$+\frac{1}{3!}f^{(3)}(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

or

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(a)(x-a)^k$$

The special case $a = 0$ is called a Maclaurin polynomial.

Outline - October 30, 2019

- **Section 3.4.5: Some Examples of Taylor Polynomial**
- **Section 3.4.8: The Error in the Taylor Polynomial Approximations**

Taylor Polynomial

Taylor Polynomial

Let a be a constant and let n be a non-negative integer. The nth degree Taylor polynomial for $f(x)$ about $x = a$ is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2$$

$$+\frac{1}{3!}f^{(3)}(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

or

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(a)(x-a)^k$$

The special case $a = 0$ is called a Maclaurin polynomial.

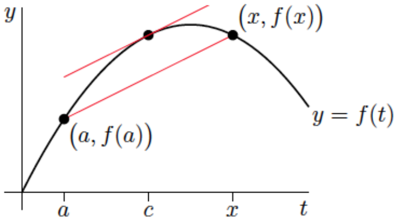
Approximating $f(x)$ by the $0th$ Taylor polynomial about $x = a$

$$f(x) \approx T_0(x) = f(a).$$

Note that

$$\begin{aligned} f(x) & \\ &= f(x) + f(a) - f(a) \\ &= f(a) + (f(x) - f(a))\frac{(x-a)}{(x-a)} \\ &= f(a) + \frac{f(x) - f(a)}{x - a}(x - a) \end{aligned} \tag{2}$$

$$f(x) = f(a) + \frac{f(x)-f(a)}{x-a}(x-a)$$



There is c strictly between x and a such that

$$f'(c) = \frac{f(x) - f(a)}{x - a}.$$

$f(x) = f(a) + f'(c)(x - a)$ for some c strictly between a and x.

$f(x) = f(a) + f'(c)(x - a)$ for some c strictly between a and x.

$$\Rightarrow f(x) - f(a) = f'(c)(x - a) \Rightarrow f(x) - T_0(x) = f'(c)(x - a)$$

The error in constant approximation

$$R_0(x) = f(x) - T_0(x) = f'(c)(x - a)$$

for some c strictly between a and x

The error in linear approximation

$$R_1(x) = f(x) - T_1(x) = \frac{1}{2} f''(c)(x-a)^2$$

for some c strictly between a and x

Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

for some c strictly between a and x

Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

for some c strictly between a and x

Remark

Consider that $f(x) = R_n(x) + T_n(x)$ Therefore,

1. if $0 \leq R_n(x) \leq E$, then

$$T_n(x) \leq f(x) \leq T_n(x) + E.$$

2. if $E \leq R_n(x) \leq 0$, then

$$T_n(x) + E \leq f(x) \leq T_n(x).$$

Outline - Nov. 1, 2019

- **Section 3.4.8: The Error in the Taylor Polynomial Approximations**
- **Section 3.5.1: Maxima and Minima**

Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

for some c strictly between a and x

Accurate to D decimal places

Generally we say that our estimate is "accurate to D decimal places" when

$$|error| < 0.5 \times 10^{-D}.$$

Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

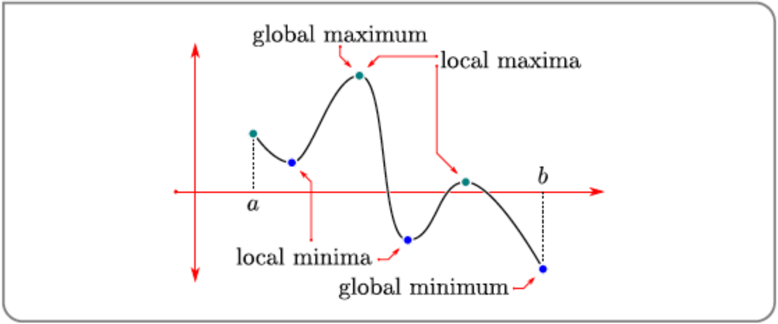
for some c strictly between a and x

Remark

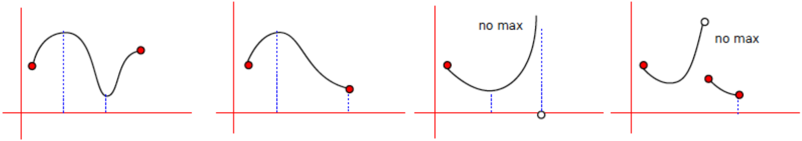
Consider that $f(x) = R_n(x) + T_n(x)$ Therefore,

1. if $0 \le R_n(x) \le E$, then
$$T_n(x) \le f(x) \le T_n(x) + E.$$
2. if $E \le R_n(x) \le 0$, then
$$T_n(x) + E \le f(x) \le T_n(x).$$

Maximum and Minimum



Continuity and global max/min



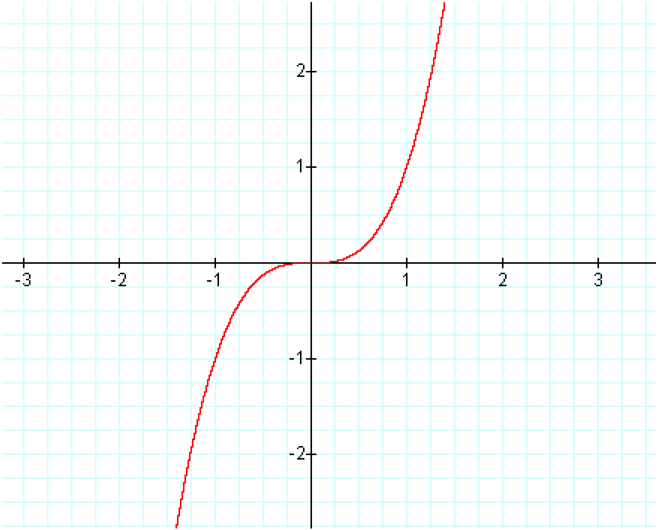
First one: Continuous/global min and max

Second one: Continuous/global min and max

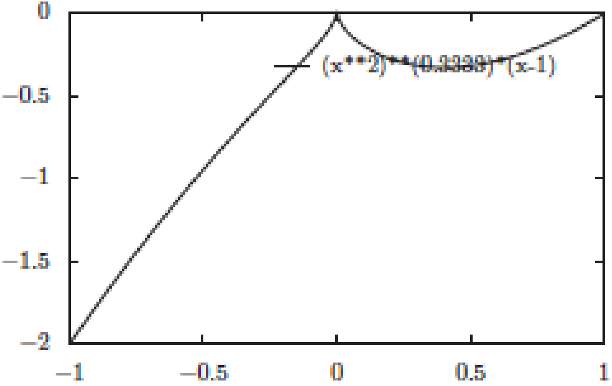
Third one: Not continuous/global min/no global max

Forth one: Not continuous/global min/no global max

If $f'(c) = 0$, then c is local max/min?!



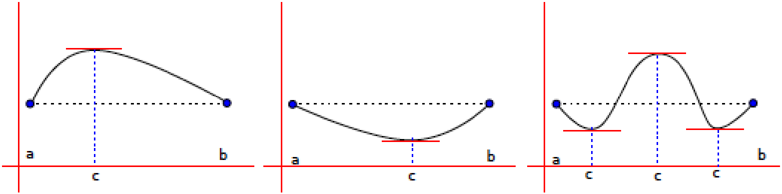
The graph of the function $x^{5/3} - x^{2/3}$ for $-1 \leq x \leq 1$



Outline - Nov. 6, 2019

- **Section 2.13: MVT**
- **Section 3.6: Sketching Graphs**

Rolle's Theorem



Rolle's Theorem

Rolle's Theorem

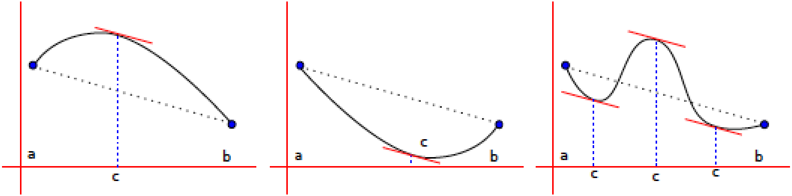
Theorem

(CLP 2.13.1–Rolle's theorem) Let f be a function such that

- *f is continuous on $[a, b]$,*
- *f is differentiable on (a, b),*
- *$f(a) = f(b)$.*

Then there is a point c between a and b so that $f'(c) = 0$.

The Mean Value Theorem (MVT)



MVT

The Mean Value Theorem

Theorem

(CLP 2.13.4–The mean value theorem) Let f be a function such that

- *f is continuous on $[a, b]$, and*
- *f is differentiable on (a, b).*

Then there is a point c between a and b so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or equivalently,

$$f(b) - f(a) = (b - a)f'(c).$$

Rolle's Theroem and IVT

Rolle's Theorem

Theorem

(CLP 2.13.1–Rolle's theorem) Let f be a function such that

- *f is continuous on $[a, b]$,*
- *f is differentiable on (a, b),*
- *$f(a) = f(b)$.*

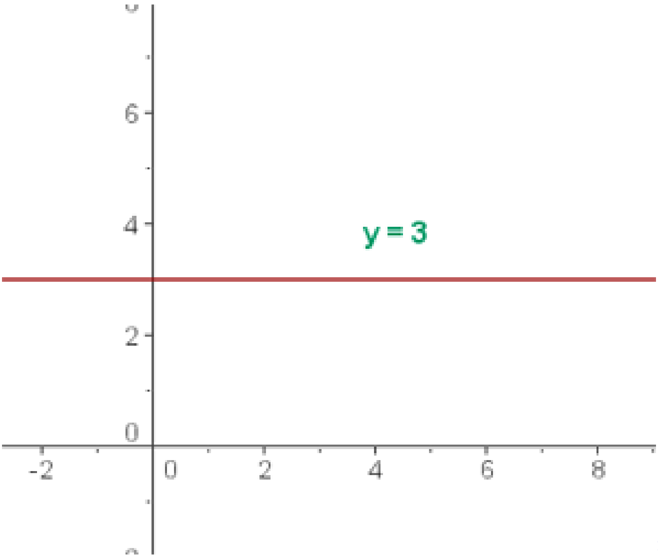
Then there is a point c between a and b so that $f'(c) = 0$.

Intermediate value theorem(IVT)

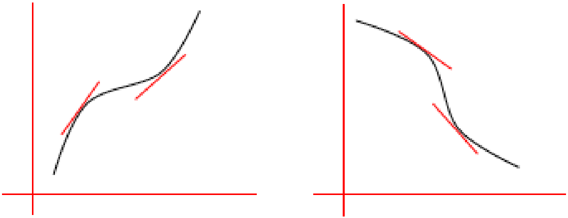
Theorem

Let $a < b$ and let $f(x)$ be a function that is continuous at all points $a \leq x \leq b$. If Y is any number between $f(a)$ and $f(b)$ then there exists some number $c \in [a, b]$ so that $f(c) = Y$.

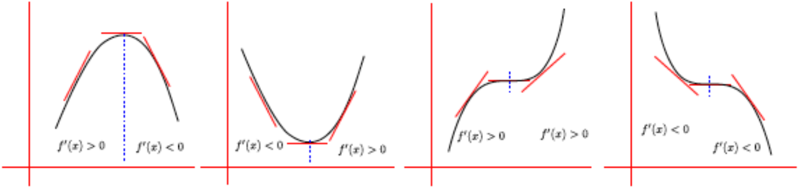
If $f'(x) = 0$ for all $x \in (a, b)$, then $f(x)$ is constant on (a, b)



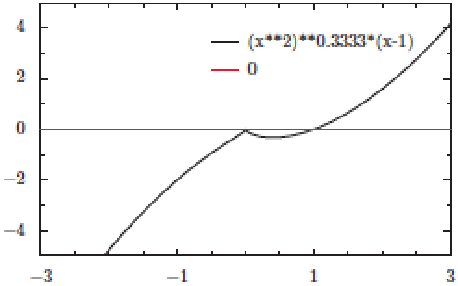
$f'(x) > 0$ then f is increasing;$f'(x) < 0$ then f is decreasing



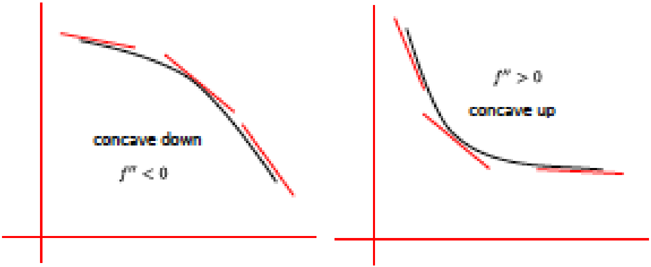
When a critical or singular point of a continuous function is a local max/min

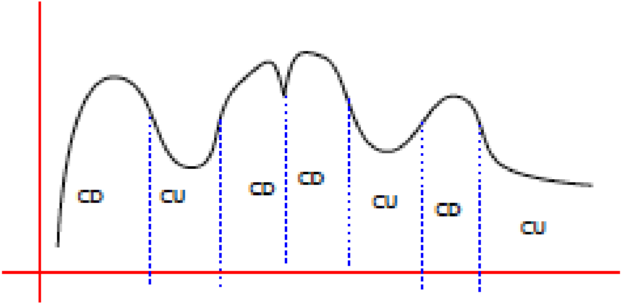


Example



Concave Up and Down





Outline - Nov. 8, 2019

- **Section 3.6: Sketching Graphs**

Different Level of Learning

- **Learning Objectives:** Be able to show that a differentiable function has exactly one or two (or more) zeros.

1. Recall and Memorize:

IVT, MVT, and Rolle's; and how we solve the question

2. Understand:

An Example by Farid

3. Apply:

Examples by You

3. Analyze:

Explain and Analyze Your Work

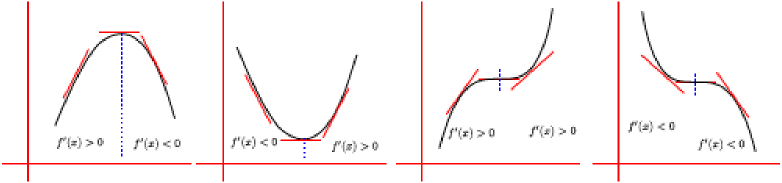
4. Evaluate:

Other's solution is Correct?

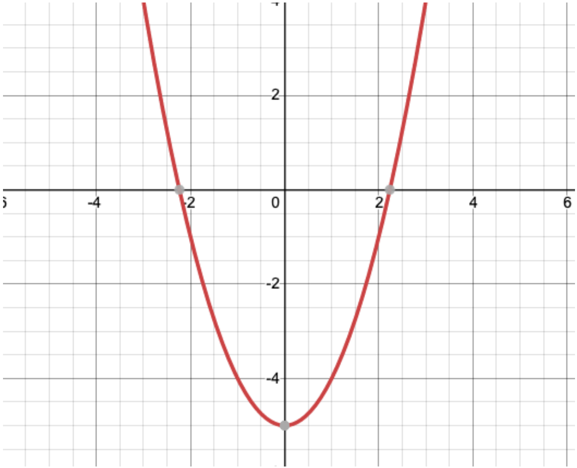
4.Create:

What questions do you give in the test?

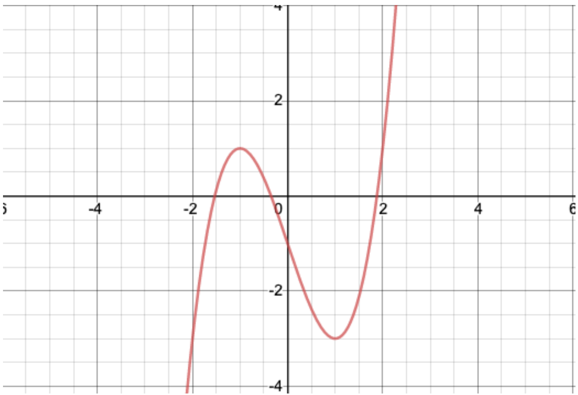
Second Derivative Test



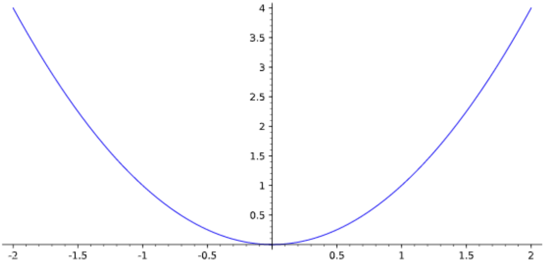
Consider the graph of $x^2 - 5$



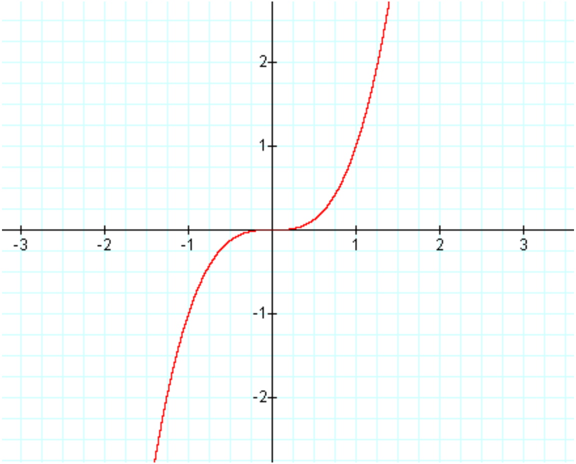
Consider the graph of $x^3 - 3x - 1$



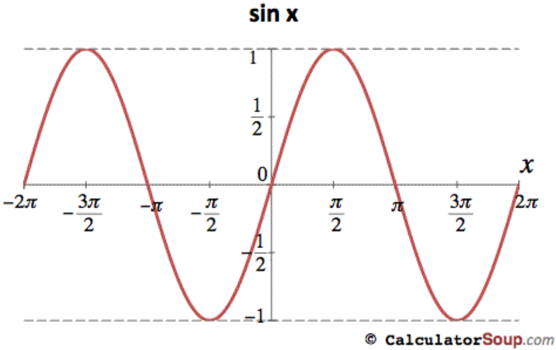
$f(x) = x^2$ is even



$f(x) = x^3$ is odd



$f(x) = \sin(x)$ is periodic



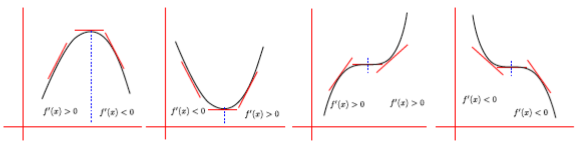
Outline - Nov. 13, 2019

- **A Quick Review**
- **Section 3.6: Sketching Graphs**

Theorem

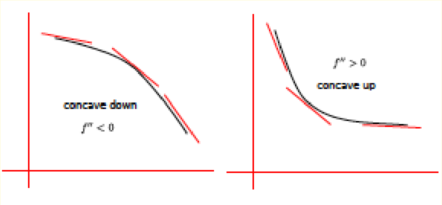
Let f be a continuous function and c be a singular or critical point. Then

- *If f' changes from positive to negative at c, then f has a local max at c.*
- *If f' changes from negative to positive at c, then f has a local min at c.*
- *If f' does not change sign at c, then c is not a local max or min.*

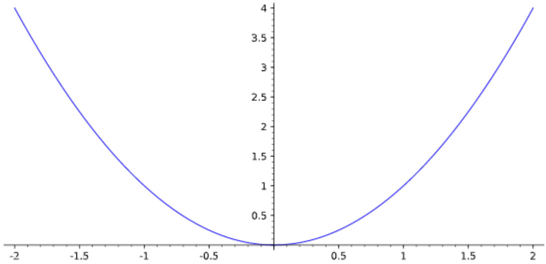


Theorem

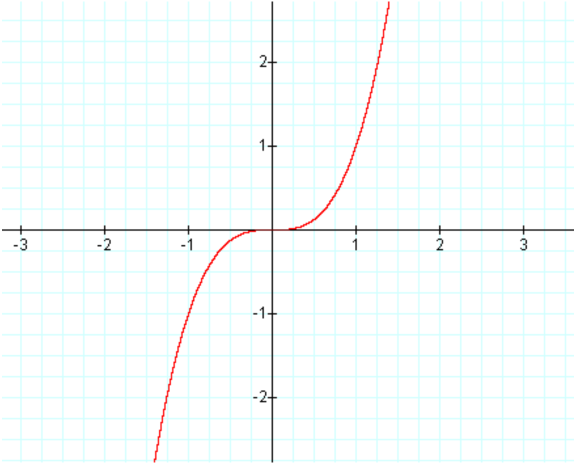
- *If $f''(x) > 0$ on I, then it is CU on I.*
- *If $f''(x) < 0$, then it is CD on I.*



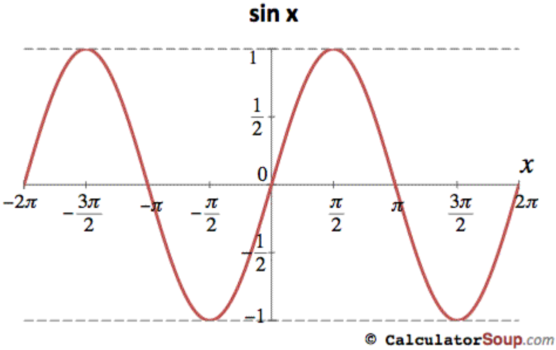
$f(x) = x^2$ is even



$f(x) = x^3$ is odd



$f(x) = \sin(x)$ is periodic



Outline - Nov. 15, 2019

- **Section 3.6: Sketching Graphs**
- **Section 3.7: Indeterminate forms and L'Hopital's rule**

$$\lim_{x\to 0}\frac{\cos(x)-1}{x}=\frac{0}{0}=??? \quad \text{or} \quad \lim_{x\to +\infty}\frac{\ln(x)}{x}=\frac{\infty}{\infty}=???$$

$$\lim_{x\to\infty}(1+\frac{3}{x})^x=1^\infty=???$$

Check-List

• **Sketching a graph.** A good check-list for sketching a graph.

- Domain
- Intercepts
- Symmetry
- Asymptotes
- Singular and critical points; Increasing/Decreasing
- Concavity and inflection points

Different Level of Learning

- **Learning Objectives:** Be able to do all steps in the check list and sketch the graph

1. Recall and Memorize:

Check-List

2. Understand:

An Example by Farid

3. Apply:

Examples by You-Knowing what theorem or method you should use

4. Analyze:

Explain and Analyze Your Work

5. Evaluate:

Other's solution is Correct?

6.Create:

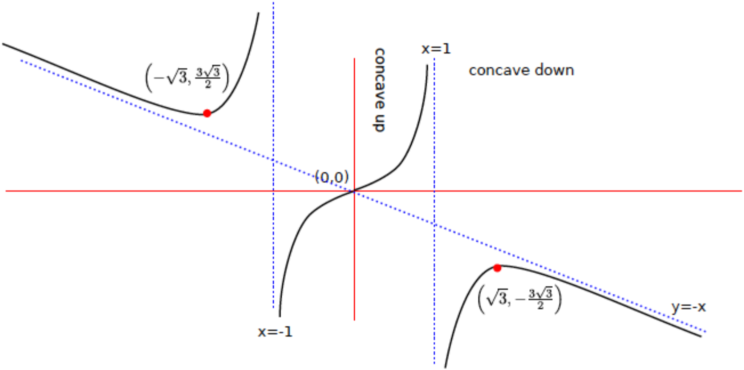
What questions do you give in the test?

$f(x) = \frac{x^3}{1-x^2}$ $f'(x) = \frac{x^2(3-x^2)}{(1-x^2)^2}$, and $f''(x) = \frac{2x(3+x^2)}{(1-x^2)^3}$.

$(-\infty, -\sqrt{3})$	$f'(x) < 0$	D	$f''(x) > 0$	CU
$x = -\sqrt{3}$	$f'(x) = 0$	$lmin$	$f''(x) > 0$	CU
$(-\sqrt{3}, -1)$	$f'(x) > 0$	I	$f''(x) > 0$	CU
$x = -1$	NE	S	NE	NE
$(-1, 0)$	$f'(x) > 0$	I	$f''(x) < 0$	CD
$x = 0$	$f'(x) = 0$	C	$f''(x) = 0$	$Inflection$
$(0, 1)$	$f'(x) > 0$	I	$f''(x) > 0$	CU
$x = 1$	NE	S	NE	NE
$(1, \sqrt{3})$	$f'(x) > 0$	I	$f''(x) < 0$	CD
$x = \sqrt{3}$	$f'(x) = 0$	$lmax$	$f''(x) < 0$	CD
$(\sqrt{3}, \infty)$	$f'(x) < 0$	D	$f''(x) < 0$	CD

Asymptotes: $x = 1$ and $x = -1$, lmax: $(\sqrt{3}, -\frac{3\sqrt{3}}{2})$ and lmin: $(-\sqrt{3}, \frac{3\sqrt{3}}{2})$. Also $f(x)$ is **odd**.

$f(x) = \frac{x^3}{1-x^2}$



Indeterminate forms and L'Hôpital's rule (CLP 3.7)

Theorem

If $\lim_{x \to a} f(x) = K$ *and* $\lim_{x \to a} g(x) = L$, *then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{K}{L} \qquad \text{provided } L \neq 0.$$

As an Example:

$$\lim_{x \to 2} \frac{x^2 - 1}{x + 1} = \frac{3}{3} = 1.$$

But what about this one

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{0}{0} = ?????$$

$$= \lim_{x\to 1} \frac{x-1}{(x-1)(x+1)}$$
$$= \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}.$$

What you can do with

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{0}{0} = ???$$

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = \frac{0}{0} = ??? \quad \text{or} \quad \lim_{x\to +\infty} \frac{\ln(x)}{x} = \frac{\infty}{\infty} = ???$$

Indeterminate,

Guillaume-Francois-Antoine Marquis de **L'Hôpital** (1661-1704)

- $\lim_{x\to 0^+} x\ln(x)$
- $\lim_{x\to\infty} x^{1/x}$
- $\lim_{x\to\infty} (1+3/x)^x$
- $\lim_{x\to\infty} \sqrt{4x^2+1} - \sqrt{x^2-3x}$

Outline - Nov. 18, 2019

- **A Quick Review**
- **Section 3.7: Indeterminate Forms and L'Hôpital's Rule**

By the end of this section you will be able to compute limits by using L'Hôpital's rule when it's needed:
(1) Change the indeterminate forms of types

$$0 \times (\pm\infty) \qquad 1^{\infty} \qquad 0^{0} \qquad \infty^{0} \qquad \infty - \infty$$

to indeterminate forms of types

$$\pm\infty / \pm\infty \qquad 0/0$$

and then use L'Hôpital's rule,
(2) when it is better doing algebra than using L'Hôpital's rule.

Indeterminate forms

$\pm\infty/\pm\infty$ and $0/0$ are two indeterminate forms. Some other types are,

$$0 \times (\pm\infty) \qquad 1^{\infty} \qquad 0^0 \qquad \infty^0 \qquad \infty - \infty$$

If we have any of the above indeterminate forms, it is more likely that we can change it to a limit that in that limit we only need to take care of a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)},$$

and make it an indeterminate form of type $\pm\infty/\pm\infty$ and $0/0$, and then we can use L'Hôpital's rule.

L'ôpital's rule 

(CLP 3.7.2—L'Hôpital's Rule)

Let f and g be differentiable functions and a either be a real number or $\pm\infty$. Furthermore, suppose that either

- $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, or
- $\lim_{x\to a} f(x) = \pm\lim_{x\to a} g(x) = \pm\infty$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that limit on the right-hand-side exists or is $\pm\infty$.

Outline - Nov. 20, 2019

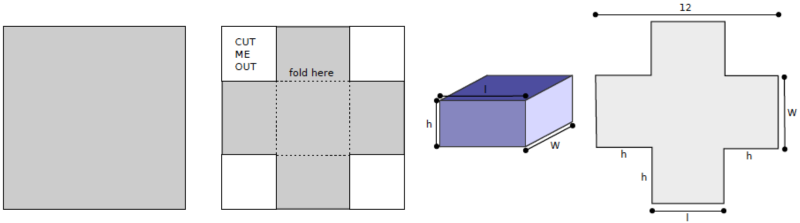
- **Section 3.5: Optimization**

By the end of this section you will be able to translate some "real world" problems to calculus and then optimizing them (finding global max/min).

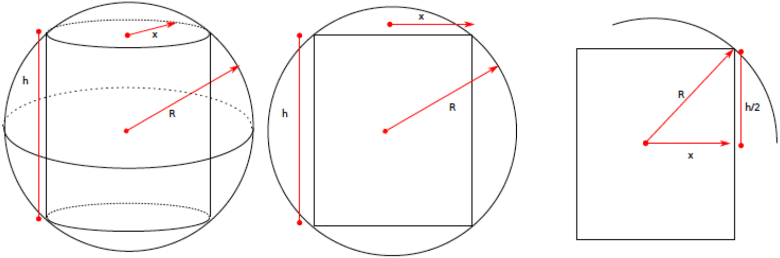
In general to answer this kind of questions, you need to

- Draw a diagram.
- Variables—assign variables to the quantities in the problem.
- Find some relation between the variables.
- Reduce to a function of 1 variable.
- Find the domain, the possible values that can be assigned to the variable.
- Max/Min: find the absolute max/min by using methods that we have studied, for example "closed interval method."

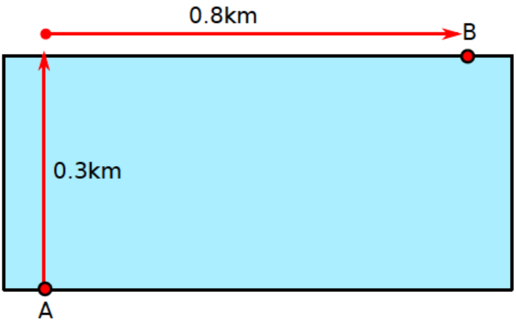
Cut-out squares and maximizing the volume



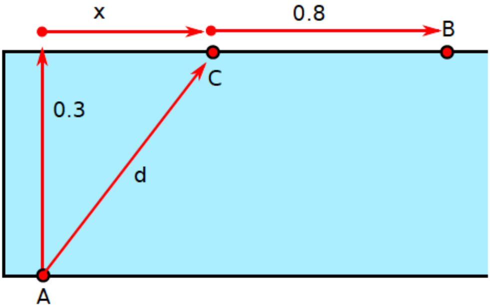
The cylinder can be inscribed a sphere



Cross a canal



Row to C, then run to B



Outline - Nov. 22, 2019

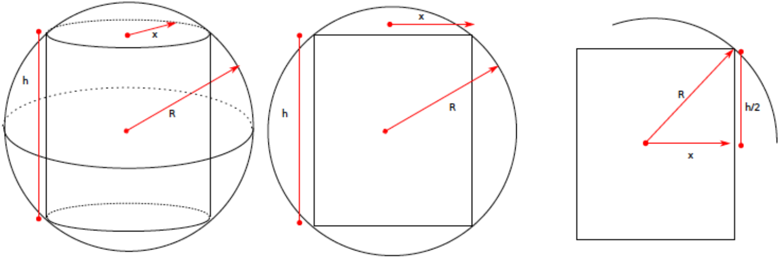
- **Section 3.5: Optimization**

By the end of this section you will be able to translate some "real world" problems to calculus and then optimizing them (finding global max/min).

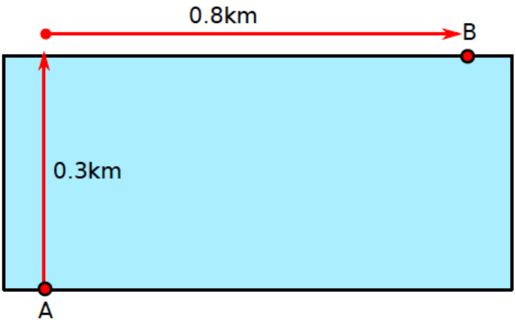
In general to answer this kind of questions, you need to

- Draw a diagram.
- Variables—assign variables to the quantities in the problem.
- Find some relation between the variables.
- Reduce to a function of 1 variable.
- Find the domain, the possible values that can be assigned to the variable.
- Max/Min: find the absolute max/min by using methods that we have studied, for example "closed interval method."

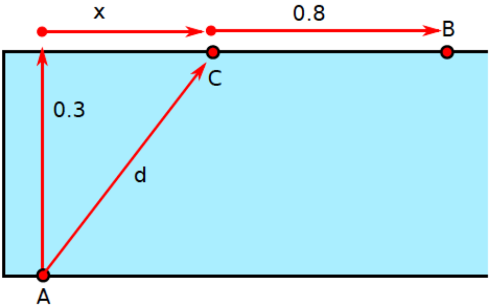
The cylinder can be inscribed a sphere



Cross a canal



Row to C, then run to B



Outline - Nov. 25, 2019

- **Section 4.1: Antiderivative**

Learning Objectives

By the end of this section,

- given a derivative $\frac{dy}{dx}$, you will be able to find what is the original function $y = f(x)$;
- you will be able to find a function $F(x)$ such that $F'(x) = f(x)$ and $F(b) = B$.

Are you here?

Pre-assessment

We have $F'(x) = 4x^3 + 1$ and $F(1) = 10$. Then

1. $F(x) = x^4 + x + 10$
2. $F(x) = 4x^4 + x + 5$
3. $F(x) = x^4 + x + 8$
4. None of the above.

Post-assessment

We have $F'(x) = 4x^3 + 1$ and $F(1) = 10$. Then

1. $F(x) = x^4 + x + 10$
2. $F(x) = 4x^4 + x + 5$
3. $F(x) = x^4 + x + 8$
4. None of the above.

Summary

- The antiderivative of a function $f(x)$ is a function $F(x)$ that $F'(x) = f(x)$; and
- the most general antiderivative is $F(x) + C$ where C is an arbitrary constant.

Pre-assessment

Find $F(x)$ if $F''(x) = 6x^2 - 18x + 14$ and $F(0) = -8$, $F(1) = -\frac{5}{2}$.

1. $F(x) = \frac{1}{2}x^4 - 3x^3 + 7x^2 - 8$
2. $F(x) = \frac{1}{2}x^4 - 3x^3 + 7x^2 + x - 8$
3. $F(x) = 2x^4 - 3x^3 + 7x^2 - 8$
4. $F(x) = 2x^4 - 3x^3 + 7x^2 + x - 8$

Post-assessment

Find $F(x)$ if $F''(x) = 6x^2 - 18x + 14$ and $F(0) = -8$, $F(1) = -\frac{5}{2}$.

1. $F(x) = \frac{1}{2}x^4 - 3x^3 + 7x^2 - 8$
2. $F(x) = \frac{1}{2}x^4 - 3x^3 + 7x^2 + x - 8$
3. $F(x) = 2x^4 - 3x^3 + 7x^2 - 8$
4. $F(x) = 2x^4 - 3x^3 + 7x^2 + x - 8$

Outline - Nov. 27, 2019

- **Review**

Learning Objectives

- Be able to compute the derivative of $f(x)^{g(x)}$.
- Be able to recall the Newton's law of cooling and use it to solve some problem.
- Be able to solve some problem regarding related rates.
- Be able to find the nth degree Taylor polynomial of some differentiable function.

Are you here?

Go to **www.menti.com** and use the code 67 47 03.

Announcements

- Your final test contains ... and ... points.
- You will be assigned a seat number.
- The previous final test probably will be sent to you soon, this test was ... and the median was ... .
- Go to Math Learning Center **MLC (Location: LSK 301 and 302)** for help
  https://www.math.ubc.ca/ MLC/

Hours of tutoring service:		
From Dec 10th till Dec 17th, 2019:	Monday - Friday	12:00pm - 6:00pm

Hours of tutoring service:		
From Sep 13th till Dec 9th, 2019:	Monday - Friday	12:00pm - 5:00pm

- My office hours: I will announce them on Friday.

Example
Go to **www.menti.com** and use the code 56 72 27

Find

$$\frac{d}{dx}x^{\sin(x)}.$$

1. $\frac{d}{dx}x^{\sin(x)} = (\ln x^{\cos(x)} + \frac{\sin(x)}{x})x^{\sin(x)}$.
2. $\frac{d}{dx}x^{\cos(x)} = (\ln x^{\sin(x)} - \frac{\cos(x)}{x})x^{\sin(x)}$.
3. $\frac{d}{dx}x^{\sin(x)} = (\ln x^{\sin(x)} + \frac{\cos(x)}{x})x^{\sin(x)}$.
4. $\frac{d}{dx}x^{\sin(x)} = (\ln x^{\sin(x)} - \frac{\cos(x)}{x})x^{\sin(x)}$.

Newton's Law of Cooling

$$\frac{dT}{dt}(t) = K\left[T(t) - A\right].$$

where $T(t)$ is the temperature of the object at time t, A is the temperature of its surroundings, and K is a constant of proportionality. Then

$$T(t) = [T(0) - A]e^{Kt} + A.$$

Example

Go to **www.menti.com** and use the code 70 06 2.

The temperature of a glass of iced tea is initially 5°. After 5 minutes, the tea has heated to 10° in a room where the air temperature is 30°.

What is the temperature after 10 minutes?

1. 11 2. 12 3. 13 4. 14

Related Rates

Go to **www.menti.com** and use the code 55 99 26.

A ball is dropped from a height of 49m above level ground. The height of the ball at time t is $h(t) = 49 - 4.9t^2$m. A light, which is also 49m above the ground, is 10m to the left of the ball's original position. As the ball descends, the shadow of the ball caused by the light moves across the ground. How fast is the shadow moving one second after the ball is dropped?

1. -100 2. -200 3. 100 4. 200

Taylor Polynomial

Let a be a constant and let n be a non-negative integer. The nth degree Taylor polynomial for $f(x)$ about $x = a$ is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 +$$

$$\frac{1}{3!}f^{(3)}(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(a)(x-a)^k$$

The special case $a = 0$ is called a Maclaurin polynomial.

Maclaurin polynomial for $\sin(x)$
Go to **www.menti.com** and use the code 33 24 27.

Example. Find the 5th degree Maclaurin polynomial for $\sin(x)$.

1. $T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$
2. $T_5(x) = x + \frac{x^3}{3!} - \frac{x^5}{5!}$
3. $T_5(x) = x + \frac{x^3}{3} - \frac{x^5}{5}$
4. $T_5(x) = 1 + \frac{x^2}{2!} - \frac{x^4}{4!}$

Lagrange remainder theorem: The error when approximating function is $T_n(x)$

$$R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

for some c strictly between a and x

Estimate ln(2)
Go to **www.menti.com** and use the code 95 98 78.

We use the third Taylor polynomial for $\ln(x)$ about $x = 1$ to estimate $\ln(2)$. Then which of the following is more accurate.
1. $|R_3(2)| \leq 1$ 2. $|R_3(2)| \leq \frac{1}{2}$
3. $|R_3(2)| \leq \frac{1}{4}$. 4. $|R_3(2)| = 0$

Outline - Nov. 29, 2019

- **Review**

Learning Objectives

- Be able to find the *n*th degree Taylor polynomial of some differentiable function and use the Lagrange Remainder Theorem.
- Be able to recall how to sketch a graph and use it to sketch the graph of a function.

Are you here?

Go to **www.menti.com** and use the code 81 18 50.

My office hours: Monday (Date: Dec 9, Time: 9-11 am), Tuesday, Wednesday (Dec 10-Dec 11) from 1:30 pm to 3:30 pm; location LSK 300.

Taylor Polynomial

Let a be a constant and let n be a non-negative integer. The nth degree Taylor polynomial for $f(x)$ about $x = a$ is

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$$\frac{1}{3!}f^{(3)}(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(a)(x-a)^k$$

The special case $a = 0$ is called a Maclaurin polynomial.

Third Taylor polynomial of $\ln(x)$
Go to **www.menti.com** and use the code 79 29 42.

Which of the following is the third Taylor polynomial of $\ln x$ about $x = 1$.

1. $1 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3$
2. $1 + (x-1) - \frac{1}{2}(x-1)^2 - \frac{2}{3!}(x-1)^3$
3. $(x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3$
4. $(x-1) - \frac{1}{2}(x-1)^2 - \frac{2}{3!}(x-1)^3$

Domain

Go to **www.menti.com** and use the code 91 54 15.

The domain of $f(x) = x(3-x)^{1/3}$ is

1. $x \leq 3$ 2. $x \geq 3$ 3. $0 \leq x \leq 3$ 4. $\mathbb{R}$.

limits

Go to **www.menti.com** and use the code 82 43 34.

Let $f(x) = x(3-x)^{1/3}$. Then $\lim_{x\to\infty} f(x) = ...$ and $\lim_{x\to\infty} f(x) = ...$

1. $-\infty, -\infty$
2. $\infty, -\infty$
3. $-\infty, \infty$
4. ∞, ∞

Derivative of $f(x)$
Go to **www.menti.com** and use the code 46 35 93.

Let $f(x) = x(3-x)^{1/3}$. Then

1. $\frac{d}{dx} f(x) = -\frac{4x-9}{3(3-x)^{2/3}}$.
2. $\frac{d}{dx} f(x) = \frac{4x-9}{3(3-x)^{2/3}}$.
3. $\frac{d}{dx} f(x) = (x-3)^{1/3} - \frac{1}{3(3-x)^{2/3}}$.
4. $\frac{d}{dx} f(x) = (x-3)^{1/3} + \frac{1}{3(3-x)^{2/3}}$.

Singular/Critical

Go to **www.menti.com** and use the code 31 06 69.

Let $f(x) = x(3-x)^{1/3}$. Then

1. $f(x)$ has a singular point at $x = 2.25$ and a critical point at $x = 3$.
2. $f(x)$ has singular points at $x = 2.25$ and $x = 3$.
3. $f(x)$ has a singular point at $x = 3$ and a critical point at $x = 2.25$.
4. $f(x)$ has critical points at $x = 2.25$ and $x = 3$.

Global max/min
Go to **www.menti.com** and use the code 23 54 74.

Let $f(x) = x(3-x)^{1/3}$. Find the global max/min (if any) of $f(x)$ on the interval $[0, 4]$

1. $f(x)$ has a global max at $x = 2.25$ and has a global min at $x = 4$.
2. $f(x)$ has a global max at $x = 4$ and has a global min at $x = 2.25$.
3. $f(x)$ has a global max at $x = 2.25$ and has no global min.
4. $f(x)$ has no global max and has a global min at $x = 4$.

Increasing/Decreasing

Go to **www.menti.com** and use the code 69 26 12.

Let $f(x) = x(3-x)^{1/3}$. Find where the function $f(x)$ is increasing and where it is decreasing.

1. $f(x)$ is increasing on $(-\infty, 2.25) \cup (3, \infty)$, and it is decreasing on $(2.25, 3)$.
2. $f(x)$ is decreasing on $(-\infty, 2.25) \cup (3, \infty)$, and it is increasing on $(2.25, 3)$.
3. $f(x)$ is decreasing on $(-\infty, 2.25)$, and it is increasing on $(2.25, \infty)$.
4. $f(x)$ is increasing on $(-\infty, 2.25)$, and it is decreasing on $(2.25, \infty)$.

Local max/min
Go to **www.menti.com** and use the code 45 33 1.

Let $f(x) = x(3-x)^{1/3}$. Find the local max/min (if any) of $f(x)$.

1. $f(x)$ has a local min at $x = 2.25$ and has a local max at $x = 3$.
2. $f(x)$ has a local max at $x = 2.25$ and has a local min at $x = 3$.
3. $f(x)$ has a local max at $x = 2.25$ and has no local min.
4. $f(x)$ has a local min at $x = 3$ and has no local max.

Second derivative

Go to **www.menti.com** and use the code 86 82 43.

Let $f(x) = x(3-x)^{1/3}$. Find the second derivative of $f(x)$.

1. $f''(x) = \frac{-4x-18}{9(3-x)^{5/3}}$
2. $f''(x) = \frac{4x+18}{9(3-x)^{5/3}}$
3. $f''(x) = \frac{-4x+18}{9(3-x)^{5/3}}$
4. $f''(x) = \frac{4x-18}{9(3-x)^{5/3}}$

Concavity

Go to **www.menti.com** and use the code 31 91 1.

Let $f(x) = x(3-x)^{1/3}$. Where $f(x)$ is concave up and where it is concave down.

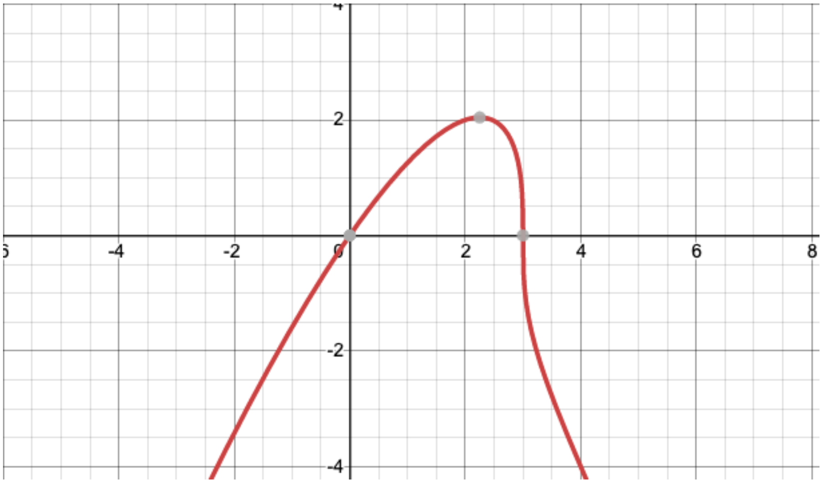
1. Concave down on $(-\infty, 4.5)$ and concave up $(4.5, \infty)$.
2. Concave down on $(-\infty, 3)$ and concave up $(3, \infty)$.
3. Concave down on $(-\infty, 3) \cup (4.5, \infty)$ and concave up $(3, 4.5)$.
4. Concave up on $(-\infty, 3) \cup (4.5, \infty)$ and concave down on $(3, 4.5)$.

Inflection points

Go to **www.menti.com** and use the code 66 59 14.

Let $f(x) = x(3-x)^{1/3}$. Find the inflection point(s) of $f(x)$.

1. The function has only one inflection point at $x = 3$.
2. The function has only one inflection point at $x = 4.5$.
3. The function has inflection points at $x = 3$ and $x = 4.5$
4. The function has no inflection points.


-4
-2
0
2
4
6
8
2
-2
-4

Go to www.menti.com and use the code 45 59 80.

Say your last words . . .