MATH 101

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2020

Learning Goals 1

(1) Approximate the area between a curve and x-axis by using **left, right, or midpoint** sums. Interpret a definite integral in terms of the area between a curve and x-axis. Compute definite integral by using the **Riemann Sum**, the definition of definite integral.

Examples,

- Estimate the area under the graph $y = \sqrt{x}$ from $x = 0$ to $x = 4$ using N approximating rectangles and right endpoints. Sketch the graph and rectangles. Is your estimate an underestimate or overestimate?
- Write an integral that is defined by the expression

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin\left(\frac{i}{4n}\right).$$

- Use the definition of a definite integral to show that

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3}.$$

Learning Goals 2

Compute definite integrals by using the **fundamental theorem of calculus**. Be able to recognize functions that are given as definite integrals with variable upper and lower limits and find their derivatives, relate antiderivatives to definite and indefinite integrals, and the **net change** as the definite integral of a rate of change. Examples:

1. Evaluate $\int_0^4 (\frac{x^2}{4} + \sqrt{x} + e^x) dx$.
2. Differentiate $\int_{\ln x}^{e^x} \frac{1}{\sqrt{1+t^4}} dt$.

Learning Goal 3

Compute the following integrals using **substitution**.

1. $\int (2x-1)e^{x^2-x}dx$.
2. $\int tan^3(\theta)d\theta$.
3. $\int_e^{e^2} \frac{1}{x \ln x}dx$.
4. $\int_2^3 \frac{1}{e^x+e^{-x}}dx$.

Learning Goal 4

Construct an integral or a sum of integrals that can be used to find the **volume of a solid** by considering its **cross-sectional areas.** For solids that are obtained by revolving a region about an axis of rotation, find the volume by considering **cross-sectional discs or washers.** Examples:

- Let R be one of the infinitely many regions bounded by $y = 1 + \sec x$ and $y = 3$. Find the volume of the solid obtained by rotating R about $y = 1$.
- Find the volume of the solid by rotating the region bounded by $y = x^2$ and $y = x + 2$ about the line $x = 3$.
- Consider a cone with base radius of r cm and height of h cm. Use the method of cross sections to show that the volume of the cone is $\frac{1}{3}\pi r^2 h \ cm^3$.
- The base of a solid S is the triangular region with vertices $(0,0),(1,0)$ and $(0,1)$. Each cross-section of S perpendicular to the y-axis is a right isosceles triangle with hypotenuse on the xy-plane. Find the volume of S.

Learning goals 5

Construct an integral or a sum of integrals that can be used to find the total work done in moving an object either by considering the entire object moving over an infinitesimal distance or by considering an infinitesimal section of the object moving over the entire distance. Examples:

- If 6 J of work is needed to stretch a spring from 10 cm to 12 cm and another 10 J is needed to stretch it from 12 cm to 14 cm, what is the natural length of the spring?
- A 10-kg bucket containing 36 kg of water is lifted from the ground to a height of 12 m with a rope that weigh 0.8 kg/m. How much work is done?
- A spherical tank of radius 5 m is built underground with the top 3 m below ground level. If the tank is full of water, how much work is needed to pump all the water out to the ground level? Assume that the density of water is 1000 kg/m^3.

Learning Goal 6

Compute integrals of basic functions by using antiderivative formulas and techniques such as **substitution and integration by parts**.

- $\int \frac{\ln(x)}{x^7} dx$
- $\int x^2 \arctan(x) dx$
- $\int \sqrt{x} e^{\sqrt{x}} dx$
- $\int \sin(\ln(x)) dx$
- Compute $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}(\frac{2}{n}i - 1)e^{(\frac{2}{n}i-1)}$ by changing it to a definite integral.

Also, **trigonometric integrals,**

- $\int \sin^a(x) \cos^b(x) dx$
- $\int \tan^a(x) \sec^b(x) dx$

Learning Goal 7

Compute the integrals of the following functions by using trig substitution.

- $\int \frac{dx}{\sqrt{3-2x-x^2}}$
- $\int \frac{dx}{(x^2+1)^3}$
- $\int \frac{1}{(2x-3)^2\sqrt{4x^2-12x+8}}$ when $x > 2$

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$
$\sqrt{a^2 + x^2}$ or $\frac{1}{a^2+x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$

Learning Goals 8: Partial Fraction

Computing the following integrals of rational functions.

- $\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$
- $\int \frac{12x+4}{(x-3)(x^2+1)}\,dx$

denominator factor	partial fraction expansion	covered in this course
$(x-a)$	$\frac{A}{x-a}$	✓
$(x-a)^r$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_r}{(x-a)^r}$	✓
(x^2+bx+c)	$\frac{Bx+C}{x^2+bx+c}$	✓
$(x^2+bx+c)^r$	$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \frac{B_3x+C_3}{(x^2+bx+c)^3} + \cdots$	× (phew)

Learning Goals 9: Numerical Integration

Approximating the value of a definite integral using the **midpoint rule**, the **trapezoidal rule**, and **Simpson's rule.** Examples:

(a) Use trapezoidal rule to approximate $\int_4^6 \ln(x^3 + 2)dx$ with $n = 10$. Estimate the error of the approximation.

(b) How large should n be to guarantee that the Simpson's rule approximation to $\int_0^1 e^{x^2} dx$ is accurate to within 1×10^{-5}?

Learning Goals 10: Improper Integral

Determine whether an **improper integral** (which either has infinite lower or upper limits of integration, or has an integrand with infinite discontinuities within or at the boundary of the interval of integration) diverges or converges, by evaluating the improper integral or by using the comparison theorem. Examples:

(a) Determine wether $\int_0^\infty \frac{dx}{x^2+3x+2}$ is divergent or convergent. If it is convergent, evaluate the integral.

(b) Find the area of the region $\{(x, y)|0 \le x \le \pi/2, 0 \le y \le \tan x\}$.

Learning Goals 11: Center of Mass

Construct an integral or a sum of integrals that can be used to find the moment about the x- or y-coordinate of the centroid of a plane region. Example,

- Find the moments and the centre of mass of a plate with shape
$$\{(x, y) : x \geq 0, y \geq x - 1, x^2 + y^2 \leq 1\}$$
if the density of the plate is 2.
- Let E be the ellipse $x^2 + k^2y^2 = 1$, where k is a constant and $k > 1$. Let S be the region inside the circle $x^2 + y^2 = 1$, outside E, and above the x-axis. Find all values of k such that the centre of mass of S lies inside S.

Learning Goals 12: Separable Equations and Sequences

Find a function that solves a **separable equation** with or without initial condition.

- Solve the initial-value problem $\frac{dy}{dx} = y^4(x+1)^2, \quad y(0) = -1$. Express your answer of the form $y = f(x)$ and simplify.

Determine if a sequence is **convergent** or **divergent** and evaluate the **limit** of a convergent sequence. Determine if a sequence is **increasing**, **decreasing**, or **bounded** .

- Does the sequence $\left\{\frac{(-1)^n n^3}{4n^3+3n^2+2n+1}\right\}_{n=0}^{\infty}$ converge? If so, what is the limit?
- Is the sequence $a_n = \frac{n}{n^2+1}$ increasing, decreasing or non-monotonic? Is the sequence bounded?

Learning Goals 13: Series

Be able to use the **sigma notation** to represent a **series**, and recognize that the sum of the series is the limit of the partial sums. Find the sum of a **geometric series** or determine that a geometric series is divergent. Be able to recognize **telescoping series** and **harmonic series**. Use the test for divergence to determine if a series can be concluded to be divergent. Examples:

(a) Evaluate $\sum_{n=1}^{\infty}(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}})$.

(b) Evaluate $\sum_{n=1}^{\infty} \arctan n$ or determine that the sum is divergent.

(c) Evaluate $\sum_{n=1}^{\infty} \frac{e^n}{3^{n+1}}$ or determine that the sum is divergent.

(d) Evaluate $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ or determine that the sum is divergent.